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On the Equations of Motion for Constrained Systems and Their Stability

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This paper presents recent advances and reviews several notions on constrained systems that have evolved over the years. It questions several of the "received (conventional) views" on both holonomic and nonholonomic systems and shows their erroneous nature in light of recent developments. By using simple illustrative examples, it draws attention to several key ideas that differ from conventional views on the behavior of constrained systems, and especially their stability. Using the explicit equation of motion for constrained systems, we show that an n degree-of-freedom unconstrained dynamical system upon which holonomic and/or nonholonomic constraints are imposed remains an n degree-of-freedom system with the initial conditions allowed by the constraints. A central result on the stability of systems using the explicit equation is developed. It encompasses holonomic and/or nonholonomic constraints, including a wide class of nonlinear nonholonomic constraints. Several commonly held notions related to constrained systems are examined and found to require refinement, alteration, or rejection.
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1 Introduction

The description of constrained motion has been a sort of holy grail to which numerous physicists, mathematicians, and engineers have contributed, starting from the very inception of the field of Analytical Dynamics (today, also sometimes referred to by many as Engineering Science). This continuous, confluent, and intense interest in the motion of constrained mechanical systems from three large scientific communities over the wide arc of the last two and a half centuries started after the publication of Lagrange's Treatise on Analytical Dynamics [1] and its forerunners authored by Euler [2]. A great deal of interest and emphasis has been placed over these long years on the dynamics of holonomically constrained systems—systems subjected to constraints, which can be reduced to relations between the configuration coordinates of a mechanical system, and could possibly also contain time explicitly. This emphasis is in no small measure due to the difficulty of obtaining the equations of motion for systems that have nonholonomic constraints, which we shall discuss shortly. The conformity in the enormous amount of literature on holonomically constrained systems has led to our having inherited today, what might best be called, a "received" or "conventional" view, which has been accepted, perhaps a bit too unquestioningly. This view holds that a holonomically constrained system is always tantamount to a lower-dimensional unconstrained system, because the constraints between the coordinates can be used to eliminate as many coordinates as there are constraints. However, the dynamics of the so-called lower-dimensional "reduced" system, from a geometric viewpoint, might no longer be described by a Euclidean metric that has no curvature. One of the aims of this paper is to take a deeper look at such systems. We show that determining these reduced equations may not be as obvious and simple as the conventional view might suggest.

On the other hand, nonholonomic constraints are described by equations that involve the generalized velocities of a dynamical system, and they cannot be integrated to obtain relations between the coordinates without knowledge of the equations of motion of the constrained system, which, of course, is what one is in quest of, in the first place, to describe the behavior of such systems. Following Lagrange and Jacobi, the conventional way to obtain the equations of motion for such a system is through the use of Lagrange multipliers (e.g., Refs. [3–10]), introduced originally by Lagrange for finding the extrema of functions on which constraints are imposed. One first finds these multipliers, and then uses them, after that, to obtain the requisite equations of motion. For systems with many degrees-of-freedom and many constraints,

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these multipliers can be very difficult to obtain. From a historical standpoint, the constraints imposed on nonholonomically constrained systems have generally been taken in the so-called Pfaffian form—that is, with equations linear in the velocities [3–10]. Accordingly, in the classical development of analytical dynamics for such systems, nonlinear terms in the generalized velocities were largely excluded, owing—at least initially—to the perceived difficulty of identifying realistic physical systems subject to such constraints. Even after Appell showed that such constraints can easily arise in real-life physical systems [11], their continual exclusion from the field of discourse appears arguably because of the difficulty in determining the Lagrange multipliers when such constraints are included and also because of the confining definition of the concept of virtual displacements, which has persisted, for such systems [3–10,12].

In this paper, we examine the equations of motion and the stability of mechanical systems subject to both holonomic and nonholonomic constraints. Several new results, obtained in recent years, are used to guide our understanding of constrained motion and to revisit and critique several of the prevailing views. New results on the stability of constrained systems are presented.

As a prelude to the development of these results, in this introductory section, we begin by describing the problem of constrained motion of systems, introducing our notation along the way; then, we present its solution. We prove that the equations of motion obtained by Lagrange, which necessarily employ Lagrange multipliers, are equivalent to the explicit equations of motion recently discovered [13–17], which do not invoke any notion of multipliers. As mentioned before, these multipliers have been perceived by mathematicians, physicists, and engineers as being problem specific and generally difficult to obtain, especially for systems with many degrees-of-freedom and several holonomic and/or nonholonomic constraints. They have been responsible, in large measure, for greatly limiting the use of Lagrange's equations to systems with just a few degrees-of-freedom and a small number of constraints. Contrary to the accepted conventional view, we prove, and obtain, in our introduction, an explicit expression, in closed form, of the Lagrange multipliers for general systems subjected to general holonomic and/or nonholonomic constraints.

The paper then develops results that revise the conventional understanding of constrained mechanical systems. The three sections that follow address holonomic systems, nonholonomic systems, and, finally, their stability. Simple examples illustrate the various issues. The conclusions point to several prevalent, and often unquestioningly accepted, conventional notions that have come through the centuries, which either need to be refined, altered, or rejected; each of these is illustrated through the examples in the text.

Consider an unconstrained n-degree-of-freedom (n-DOF) mechanical system described by the (real) generalized coordinates $q_1, q_2, \ldots, q_n$ whose motion is described by the equations

$$M(q(t), t)\ddot{q} = Q(q(t), \dot{q}(t), t),\ q(0) = q_0,\ \dot{q}(0) = \dot{q}_0 \qquad (1)$$

where the n-vector (n by 1 column vector) $q(t) = [q_1(t), q_2(t) \ldots, q_n(t)]^T$, $M(q(t), t) = M(q(t), t)^T$, $M(q(t), t) > 0$, $Q(q(t), \dot{q}(t), t)$ is the n-vector of the "known" generalized *impressed* force on the system, and the dots refer to differentiation with respect to time t. The initial conditions on the position, $q(t)$, and on the generalized velocity, $\dot{q}(t)$, of the system are prescribed at time $t = 0$ by the n-vectors q_0 and $\dot{q}_0$, respectively. By "unconstrained" we mean that the virtual displacements δq_i, $i = 1, 2, \ldots, n$, are all arbitrary, and the $2n$ (finite) entries in each of the n-vectors q_0 and $\dot{q}_0$ can be *arbitrarily* chosen. The acceleration of the unconstrained system (1) will be denoted by $a(q(t), \dot{q}(t), t) := M(q(t), t)^{-1}Q(q(t), \dot{q}(t), t)$. Here, by "known" we mean that $Q(q(t), \dot{q}(t), t)$ is a known function of its arguments $q(t), \dot{q}(t)$, and t. We note that Lagrange's equations of motion for mechanical systems in classical mechanics always take the form shown in Eq. (1).

We assume next that the unconstrained n-DOF system described in Eq. (1), which is subjected to the impressed force Q, is now subjected to the $m = h + u$ consistent constraints described by the equations

$$\varphi_i(q(t), t) = 0,\ i = 1, \ldots, h \qquad (2)$$

$$\psi_j(q(t), \dot{q}(t), t) = 0,\ j = 1, \ldots, u \qquad (3)$$

where the h functions $\varphi_i \in C^2$, and the u functions $\psi_j \in C^1$. Furthermore, the u functions ψ_j in Eq. (3) are nonintegrable, that is, they cannot be integrated with respect to time t, and placed in the form shown in Eq. (2) before obtaining the equations of motion of the constrained system. As mentioned before, the set of constraints of the form (2) are called holonomic, while the set of the form (3) are called nonholonomic. The former restricts the positions, $q(t)$, and the velocities, $\dot{q}(t)$, of the system, the latter restricts a combination of the positions and velocities of the system. In what follows, for brevity, we will drop the arguments of $q(t)$ and $\dot{q}(t)$.

Unconstrained mechanical systems subjected only to constraints of the form given in Eq. (2) are called holonomic systems; unconstrained mechanical systems subjected only to constraints of the form given by Eq. (3) are often called nonholonomic systems; those that have constraints of both forms, Eqs. (2) and (3), will be called in this article general constrained systems. Some authors refer to general dynamical systems as those that have at least one nonholonomic constraint, among possibly many. Though widely prevalent, the notion that the presence of a single nonholonomic constraint among holonomic ones always yields a nonholonomic system is incorrect, as will be shown later.

Simple examples of holonomic constraints are $q_1 = c_1 q_2^2 + c_2 q_3^2 + c_3 \sin(t)$ and $q_1^2 + c_1 q_2^2 + c_2 q_3^2 = c_3^2$; examples of nonholonomic constraints are $\dot{q}_1 = c_1 q_2 \dot{q}_3 + c_2 q_3 + c_3 \cos(t)$ and $\dot{q}_3^2 = c_1 q_1^2 + c_2 \dot{q}_2^2 + \tanh(t)$. Here, the $c_i's$ are real numbers. Often, a general constrained mechanical system is considered as having a combination of these constraints—at least one of each type—imposed on it.

An important observation is that, in the presence of constraints, the components of the n-vectors q_0 and $\dot{q}_0$ that specify the initial conditions of the constrained system can no longer be chosen arbitrarily, as in the unconstrained system (1), because they must satisfy each of the constraints in Eq. (2) and/or Eq. (3), as appropriate, at time $t = 0$.

It should be noted that the equations in the set of nonholonomic constraints are actually differential equations, and though they cannot be integrated (without first knowing the equations of motion of the constrained system) they can be differentiated, assuming that they are C^1. Differentiating each of the h equations in the set (2) twice with respect to time t, and each of the equations in the set (3) once with respect to time, we obtain the set of m constraint equations given by

$$A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t) \qquad (4)$$

where $A(q, \dot{q}, t)$ is an m by n matrix and $b(q, \dot{q}, t)$ is an m-vector. Each row in Eq. (4) corresponds to the differentiated form of each constraint. We shall, as before, drop the arguments of A and b in what follows, and refer to the matrix A as the constraint matrix. For brevity, we will call Eq. (4) the "constraint equation." Virtual displacements are nonzero (column) n-vectors that lie in the null space of the constraint matrix A, i.e., n-vectors $v_i(t)$ such that $Av_i = 0$ [12]. Contrary to the conventional view, these vectors need not have infinitesimal elements, since any vector v_i that satisfies this equation guarantees that the vector αv_i also does, for arbitrary (nonzero) finite values of α.

The above framework provides the precise setting for the problem of constrained motion in mechanical systems, and is essential to understanding their dynamical behavior.

1.1 Explicit Equation of Motion for Constrained Systems. Having described a constrained mechanical system, we now go on to give the equation of motion of the unconstrained n-DOF system (1) when the m constraints given by Eqs. (2) and (3) are imposed on it, or alternatively, in the presence of the constraint equation (4). By "given" we mean here that the elements in the m by n matrix A and in the m-vector b are known functions of $q, \dot{q}$, and t.

As already noted, the problem of describing the motion of such systems in a unified manner has long engaged both physicists and engineers, each community having encountered innumerable examples when idealizing naturally occurring and engineered systems through mathematical models. Over more than two centuries, mathematicians too have been deeply concerned with the description of constrained motion, particularly from the standpoint of differential equations. To date, some of the conventional views held by many physicists, mathematicians, and engineers are that:

(1) Holonomic systems are tantamount to unconstrained systems because they can always be "reduced" to unconstrained systems; they are easier to handle, and they have isolated equilibrium states,
(2) For nonholonomic systems, an explicit, general equation of motion cannot be found; each specific mechanical system must be considered individually, and
(3) Nonholonomic systems do not possess isolated equilibrium states; their equilibria are always nonisolated and are responsible for the difficulties in analyzing their stability.

By nonisolated equilibrium states, above is meant a continuum of equilibrium positions in phase space. Regarding holonomic systems, the conventional view is that for an unconstrained system with n degrees-of-freedom upon which m independent holonomic constraints are imposed, a (local) coordinate transformation can always "reduce" the system to one that has $(n-m)$ (independent) degrees-of-freedom. Thus, a system with a reduced number of degrees-of-freedom, namely, an $(n-m)$ degree-of-freedom system, is essentially obtained. In the conventional view, an unconstrained n-DOF system upon which m such holonomic constraints are imposed is tantamount to, and is simply taken to be, an $(n-m)$ degree-of-freedom system (see, for example, Ref. [4]).

Regarding nonholonomic systems, the book by Goldstein et al. [7], for decades a gold-standard in classical mechanics among physicists, states up front the conventional view that, "But there is no general way to attack nonholonomic examples.… the more vicious cases of nonholonomic constraint must be tackled individually, and consequently in the development of the more formal aspects of classical mechanics it is invariably assumed that the constraint, if present, is holonomic." Most physics-oriented books do not deal with nonholonomic constraints in any significant depth.

In contradistinction to this general, somewhat pessimistic, background, summarized from work done by numerous researchers across more than two centuries, we now present in brief the determination of the equation of motion for an unconstrained n-DOF system described in Eq. (1), subjected to the $m = h + u$ consistent constraints that are holonomic and/or nonholonomic as in Eqs. (2) and (3). The initial conditions for the constrained system are such that at $t = 0$ all the constraints are satisfied.

We note that the constrained system is fully described by the n by n mass matrix $M > 0$, the m by n constraint matrix A, the acceleration n-vector a of the unconstrained system, the impressed force n-vector Q on the n-DOF unconstrained system, and the m-vector b created in the constraint equation (4). Suppressing the arguments in these entities, under the assumption that d'Alembert's principle is satisfied by the force of constraint Q^c, the equation of motion of the constrained system is explicitly given by [13–17]

$$\begin{aligned} M\ddot{q} &= Q + Q^c := Q + A^T(AM^{-1}A^T)^+(b - Aa) \\ &= Q + M^{1/2}(AM^{-1/2})^+(b - Aa),\ q(0) = q_0,\ \dot{q}(0) = \dot{q}_0 \end{aligned} \tag{5}$$

where X^+ denotes the (unique) Moore–Penrose inverse of the matrix X [18], and the n-vectors q_0 and $\dot{q}_0$ giving the initial conditions have elements that are *no longer arbitrary* now, but *satisfy* the $m = h + u$ consistent constraints at time $t = 0$ that are imposed on the system as given in Eqs. (2) and (3). Note that Eq. (5) is valid for holonomic systems, nonholonomic systems, and general constrained systems.

The dynamical behavior of the constrained mechanical system is captured explicitly in Eq. (5). The constrained system has n degrees-of-freedom (n-DOF), the same number as the original unconstrained system (1). Comparing Eq. (5) with Eq. (1), the term Q^c on the right-hand side of Eq. (5) encapsulates the total effect of all imposed constraints on the unconstrained n-DOF system. It modifies the system's response to the impressed force by adding an additional force, ensuring that the constrained n-DOF system exactly satisfies the constraints when started at time $t = 0$ from any set of initial conditions q_0 and $\dot{q}_0$ that satisfy the imposed constraints.

From a "differential equations" standpoint, this second member on the right-hand side of Eq. (5) that gets added to the impressed force Q that the unconstrained n-DOF system is subjected to, causes all the constraints to be exactly satisfied for $t > 0$, thus making each imposed constraint an integral of motion of the constrained n-dimensional system described by Eq. (5), when the constrained system is started from any set of initial conditions that are consistent with the constraints. Each constraint φ_i and ψ_j in Eq. (5) is a *conserved quantity* during the motion of the constrained system.

It should be noted that the explicit equation of motion in Eq. (5) has no Lagrange multipliers. In fact, in the derivation of Eq. (5) no appeal is made to any notion of a Lagrange multiplier [13–17]. As mentioned before, in systems with even a small number of degrees-of-freedom ($n \sim 5$–15) and a few ($u \sim 3$–7) nonholonomic constraints, Lagrange multipliers are often difficult to obtain. Engineered mechanical systems, as well as naturally occurring ones, are often modeled with hundreds (if not thousands) of degrees-of-freedom, and with numerous holonomic and nonholonomic constraints, generally making the determination of all the Lagrange multipliers extremely difficult, if not arguably impossible.

For over two centuries, Lagrange multipliers have been widely used to derive the equations of motion for constrained mechanical systems. Besides bypassing the very notion of Lagrange multipliers espoused in the conventional view of constrained motion [3–10], and therefore the need for finding expressions for them, Eq. (5) is valid when: (i) the u nonholonomic constraints $\psi_j(q, \dot{q}, t) = 0$ can have nonlinear terms in q, $\dot{q}$, and t; (ii) one or more of the imposed constraints are repeated any number of times in the list of constraints imposed, and the constraints need not be independent of one another, as always necessitated by the conventional view [3–10]; (iii) the constraints are holonomic, nonholonomic, or a combination of holonomic and nonholonomic; and (iv) one desires to assess the dynamical influence of the addition or removal of one or more constraints on the system. This can be easily done because it involves simply the addition or removal of one or more of the appropriate rows in the m by n constraint matrix A and correspondingly in the column m-vector b; holonomic constraints can be added/removed as easily as nonholonomic ones. Item (ii) above is useful because in the modeling of complex multidegree-of-freedom systems, the nonholonomic constraints being differential equations, they may acquire a different form when using a multiplier, making it difficult to determine whether the constraints are, indeed, independent.

If Lagrange multipliers are needed, as demanded by the conventional and most prevalent view [3–10], which requires that they be determined first, so that the equation of motion of the constrained system can then be obtained by using them, we determine explicitly, and in closed form, the Lagrange multipliers for mechanical systems that are holonomically and/or nonholonomically constrained. We start by showing that Lagrange's equation of motion is equivalent to that given in Eq. (5).

1.2 Equivalence of Lagrange's Equation for Constrained Motion and the Explicit Fundamental Equation (5).

Lagrange's equation expresses the motion of a constrained system as ($M > 0$ is an n by n symmetric matrix)

$$M\ddot{q} = Q + A^T\lambda \tag{6}$$

where the m-vector, λ, is the Lagrange multiplier, which needs to be determined using Eq. (6) *and* the $m = h + u$ constraints given in Eqs. (2) and (3). Using the second equality on the right-hand side in Eq. (5) and subtracting Eq. (5) from Eq. (6), we get

$$A^T\lambda = M^{1/2}(AM^{-1/2})^+(b - Aa) := M^{1/2}(AM^{-1/2})^+z \tag{7}$$

We now show that Eq. (7) is consistent. That is, for *any* m-vectors z, there exists at least one m-vector λ such that Eq. (7) is true. Premultiplying both sides of Eq. (7) by $M^{-1/2}$ gives

$$B^T\lambda = B^+z \tag{8}$$

where we have denoted $B := AM^{-1/2}$. A necessary and sufficient condition for Eq. (8) to be consistent is that [14]

$$B^T(B^T)^+B^+z = B^+z$$

which is always true, since $B^T(B^T)^+B^+ = B^T(B^+)^TB^+ = (B^+B)^TB^+ = B^+BB^+ = B^+$. This proves that Eq. (8) is consistent and can be solved for the m-vector, λ, for any given m-vector, z. An alternative way of showing that Eq. (8) is consistent is by observing that the range space of B^T is the same as that of B^+ [14].

1.3 Explicit Determination of the Lagrange Multiplier *m*-Vector.

If needed, λ can be explicitly solved from Eq. (7) to yield [17]

$$\lambda = (A^T)^+M^{1/2}(AM^{-1/2})^+(b - Aa) + [I - (A^T)^+A^T]w$$

where w is an *arbitrary* m-vector. Noting that $(A^T)^+A^T = (A^+)^TA^T = (AA^+)^T = AA^+$, we get the explicit equation for the Lagrange multiplier m-vector, λ, given by [14]

$$\lambda = (A^T)^+M^{1/2}(AM^{-1/2})^+(b - Aa) + [I - AA^+]w \tag{9}$$

Since w is an arbitrary m-vector, the Lagrange multiplier λ is, in general, *not* unique. The necessary and sufficient condition for the second member on the right-hand side of Eq. (9) to vanish is $\operatorname{rank}(A) = m$, i.e., when the constraints are independent, and the m rows of the constraint matrix A are linearly independent [14]. In this case, the explicit expression for the Lagrange multiplier simplifies further, which we next obtain.

Using the first equality on the right-hand side of Eq. (5) in Eq. (7), we have

$$A^T\lambda = A^T(AM^{-1}A^T)^+(b - Aa)$$

Premultiplying both sides by the m by n matrix A yields

$$AA^T\lambda = AA^T(AM^{-1}A^T)^+(b - Aa)$$

Since $\operatorname{rank}(A) = \operatorname{rank}(AA^T) = m$, the m by m matrix AA^T is invertible. Upon premultiplying both sides by $(AA^T)^{-1}$, the explicit expression for the Lagrange multiplier m-vector when $\operatorname{rank}(A) = m$ is then given by

$$\lambda = (AM^{-1}A^T)^{-1}(b - Aa) \tag{10}$$

because $AM^{-1}A^T$ is invertible, as $M > 0$.

In the following three sections, we consider simple examples using unconstrained systems with just two or three degrees-of-freedom upon which only one or two constraints are imposed. Our purpose is to use the minimal amount of algebra to explicitly work out and get a detailed understanding of the behavior of constrained motion, so that we can illustrate conventional views, examine them, identify discrepancies, propose modifications/rejections, and highlight the remaining gaps in our understanding.

The next section deals with holonomic systems, the one that follows deals with nonholonomic systems, and Sec. 5 deals with the stability of constrained systems, in which we develop a new stability result. In our concluding section, we summarize what each of the examples reveals about (i) our current understanding, (ii) prevalent misconceptions, and (iii) the holes in our present understanding that need to be addressed.

2 Holonomically Constrained Systems

As stated before, by holonomically constrained, we mean, in general, an n-DOF unconstrained system (1), on which m consistent constraints ($m = h$) of the form (2) are imposed, resulting in an n-DOF constrained system given in Eq. (5), along with a modification of the initial conditions of the unconstrained n-DOF system so that the initial condition n-vectors q_0 and $\dot{q}_0$ of the constrained system satisfy all the imposed constraints. We illustrate and apply these notions to three holonomically constrained systems in the following simple examples. The first illustrates the "conventional view" and establishes our nomenclature; the second and third show that this view needs serious revision.

Example 1.

Consider the unconstrained 2-DOF dynamical system described by the equations

$$\ddot{x} := \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = -\begin{bmatrix} k_1x_1 \\ k_2x_2 \end{bmatrix} := Q,\ x(0) := \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} := x_0,$$
$$\dot{x}(0) := \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} := \dot{x}_0 \tag{11}$$

where the constants k_1, k_2, along with the four c_i, $i = 1, 2, 3, 4$, are each arbitrary (real) numbers. One can think of the description in Eq. (11) as being composed of: (1) the differential equation that governs the time evolution of the system, and (2) the initial conditions from which the system starts.

On this unconstrained system, we now impose the holonomic constraint $\varphi_1(x_1(t), x_2(t), t) = 0$ given by

$$\varphi_1(x_1, x_2, t) := x_1 - \frac{1}{2}x_2^2 - g\sin(t) \tag{12}$$

where g is a given (real) number.

We now ask the following question: Given the unconstrained 2-DOF system described in Eq. (11), what should the equation of motion of this 2-DOF system be if the constraint (12) is imposed on it. In short, how should the system described in Eq. (11) have to change to accommodate the constraint (12) imposed upon it?

The short answer is: both the initial conditions and the differential equation describing the time evolution of the unconstrained system would need to be changed in Eq. (11).

We begin by considering how the initial conditions need to be changed. Upon differentiating constraint (12) with respect to time t we get the relation $\tilde{\varphi}_1(x_1, x_2, \dot{x}_1, \dot{x}_2) = 0$, where

$$\tilde{\varphi}_1(x_1, x_2, \dot{x}_1, \dot{x}_2) := \dot{x}_1 - x_2\dot{x}_2 - g\cos(t) \tag{13}$$

We notice that the holonomic constraint (12) not only imposes a constraint on the positions of the system, namely, x_1 and x_2 $\forall t$, but also on its velocities, $\dot{x}_1$ and $\dot{x}_2$ $\forall t$.

Our constrained system can no longer contain arbitrary real numbers for all the c_i, $i = 1, 2, 3, 4$, as we previously had in Eq. (11), because the constraints (12) and (13) also now need to be satisfied at every time t, so also at the initial time $t = 0$. From Eq. (12), since $\left[x_1 - \frac{1}{2}x_2^2 - g\sin(t)\right]\big|_{t=0} = 0$, we must have $c_1 = \frac{1}{2}c_2^2$; similarly, from Eq. (13), $c_3 - c_2c_4 = g$. Hence, the initial conditions in Eq. (11) need to be replaced by

$$x(0) = [c_2^2/2,\, c_2]^T, \quad \text{and } \dot{x}(0) = [g + c_2c_4,\, c_4]^T \tag{14}$$

when dealing with the constrained system, and now only c_2, c_4 can be arbitrarily chosen as two (real) numbers.

Our next task is to determine how the differential equation in Eq. (11) that describes the time evolution of the unconstrained system should change to accommodate the imposed constraint (12).

We begin by differentiating Eq. (13) with respect time t (note that this is the same as differentiating Eq. (12) twice with respect to time t) to obtain

$$\ddot{x}_1 - x_2\ddot{x}_2 - \dot{x}_2^2 + g\sin(t) = 0 \tag{15}$$

which can be rearranged to give the constraint equation

$$\underbrace{[1 \quad -x_2]}_{A}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \underbrace{\dot{x}_2^2 - g\sin(t)}_{b} \tag{16}$$

in which the entities A and b are shown (in our notation, see Eq. (4)).

Next, using Eq. (5), and noting that $M = I_2$, $A^T(AA^T)^+ = A^+$, and $Q = a = -[k_1x_1, k_2x_2]^T$, we get

$$\ddot{x} = a + A^+(b - Aa) = -[k_1x_1, k_2x_2]^T + A^+(\dot{x}_2^2 - g\sin(t) + k_1x_1 - k_2x_2^2) \tag{17}$$

as the equation of motion of the constrained 2-DOF system.

Since $A^+ = \dfrac{1}{1+x_2^2}[1,\, -x_2]^T$, Eq. (17) along with Eq. (14) can now be written as:

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = -\begin{bmatrix}k_1x_1\\ k_2x_2\end{bmatrix} + \frac{1}{1+x_2^2}\begin{bmatrix}(\dot{x}_2^2 - g\sin(t) + k_1x_1 - k_2x_2^2)\\ -x_2(\dot{x}_2^2 - g\sin(t) + k_1x_1 - k_2x_2^2)\end{bmatrix},$$
$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}c_2^2/2\\ c_2\end{bmatrix}, \begin{bmatrix}\dot{x}_1(0)\\ \dot{x}_2(0)\end{bmatrix} = \begin{bmatrix}g + c_2c_4\\ c_4\end{bmatrix} \tag{18}$$

This is then the equation of motion of the constrained system. Note that Eq. (18) is a 2-DOF system, just like the unconstrained equation of motion (11) is.

Having obtained the explicit equation of motion (18) of the constrained system in the coordinates (x_1, x_2) by using Eq. (5), we now examine the conventional view that is currently prevalent.

We know that a change of coordinates can often simplify a set of ordinary differential equations. We contemplate a one-to-one (local) change in coordinates (mapping) from a sufficiently small domain D_x of (x_1, x_2, t) to a small domain D_q of a new coordinate space, (q_1, q_2, t). Since the mapping must be (locally) one-to-one, the coordinates $(q_1(x_1, x_2, t), q_2(x_1, x_2, t))$ must be chosen judiciously so that the determinant of the Jacobian, $J = \dfrac{\partial(q_1, q_2)}{\partial(x_1, x_2)}$, is not equal to zero everywhere in the local patch D_x.

Since the system described by Eq. (18) always satisfies the constraint $\varphi_1(x_1, x_2, t) = 0$, we choose the coordinate

$$q_1 = \varphi_1(x_1, x_2, t) = x_1 - \frac{1}{2}x_2^2 - g\sin(t) \tag{19}$$

so that during the evolution of the dynamical system in *any* domain D_q of coordinate space (q_1, q_2, t), we have $q_1(t) \equiv 0$, $\forall t$. Our coordinate mapping is thus from $(x_1, x_2) \to (q_1, q_2) := (\varphi_1(x_1, x_2, t), q_2)$, and our only task now is to find a suitable coordinate $q_2(x_1, x_2)$ such that the determinant of J is not zero. If we *choose* to take

$$q_2 = x_2 \tag{20}$$

then, using Eq. (19), we get

$$J := \det\begin{bmatrix}\partial q_1/\partial x_1 & \partial q_1/\partial x_2\\ \partial q_2/\partial x_1 & \partial q_2/\partial x_2\end{bmatrix} = \det\begin{bmatrix}1 & -x_2\\ 0 & 1\end{bmatrix} = 1 \tag{21}$$

Though we can only, in general, do this for a local domain D_x, we find, in this example, that this coordinate transformation works globally, since the coordinate $q_2 = x_2$ provides a Jacobian whose determinant is nowhere zero in the entire (x_1, x_2) plane, and always equals unity. Note that the coordinates q_1 and q_2 are independent. Since $q_1(t) \equiv 0$, $\forall t$, it is immediate that the evolution of the coordinate $q_1(t)$ of the constrained dynamical system is described by the equation $q(t) = \dot{q}(t) = \ddot{q}(t) = 0$, $\forall t$, and we therefore need worry only about the time-evolution of the coordinate q_2 of the constrained system.

We note that from Eq. (19) that

$$x_1(t) = q_1(t) + \frac{1}{2}x_2^2(t) + g\sin(t) = q_1(t) + \frac{1}{2}q_2^2(t) + g\sin(t)$$
$$= \frac{1}{2}q_2^2(t) + g\sin(t) \tag{22}$$

in which the second equality follows because of our choice $q_2(t) = x_2(t)$ (see Eq. (20)), and the last equality because $q_1(t) \equiv 0$. Similarly, Eq. (13) yields

$$\dot{x}_1 = q_2\dot{q}_2 + g\cos(t) \tag{23}$$

as can also be seen by simply differentiating Eq. (22). Noting that $q_2(t) = x_2(t)$, the lower differential equation in Eq. (18) then gives

$$\ddot{q}_2 = -k_2q_2 - \frac{q_2}{1+q_2^2}[\dot{q}_2^2 - g\sin(t) + k_1(\frac{1}{2}q_2^2 + g\sin(t)) - k_2q_2^2]$$

where we have substituted for x_1 from Eq. (22). This simplifies to

$$(1 + q_2^2)\ddot{q}_2 + q_2\dot{q}_2^2 + k_2q_2 + k_1q_2(\frac{1}{2}q_2^2 + g\sin(t)) - gq_2\sin(t) = 0 \tag{24}$$

The 2-DOF constrained system, whose description in the coordinates (x_1, x_2) was obtained in Eq. (18), transforms, using the invertible mapping $(x_1, x_2, t) \to (q_1(x_1, x_2, t), q_2(x_1, x_2, t), t) = (0, x_2)$, to the 2-DOF system

$$\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} = -\frac{1}{(1+q_2^2)}\begin{bmatrix}0\\ q_2\dot{q}_2^2 + k_2q_2 + k_1q_2(\frac{1}{2}q_2^2 + g\sin(t)) - gq_2\sin(t)\end{bmatrix}, \begin{bmatrix}q_1(0)\\ q_2(0)\end{bmatrix} = \begin{bmatrix}0\\ c_2\end{bmatrix}\begin{bmatrix}\dot{q}_1(0)\\ \dot{q}_2(0)\end{bmatrix} = \begin{bmatrix}0\\ c_4\end{bmatrix} \tag{25}$$

when described in the (q_1, q_2) coordinates. The first equation above is trivial; the dynamics is encapsulated in the second equation. Recall that c_2 and c_4 could be arbitrarily prescribed. This illustrates that the second equation in Eq. (25) is an equation of motion of an unconstrained system. This is proved for a general holonomically constrained system in Appendix A. Note that the evolution of $q_1(t)$ is independent of the evolution of $q_2(t)$, and the first differential equation in $q_1(t)$ is unstable at $q_1(0)=\dot{q}_1(0)=0$, since an arbitrarily small initial perturbation in $\dot{q}_1$ will cause $q_1(t)$ to become unbounded. We will see later that this observation carries significant implications from a computational standpoint.

One can see the unconstrained nature of the second equation more clearly as follows. From Eqs. (22) and (20), observe that the coordinates x_1 and x_2 are both functions of only the coordinates q_2 and t. Using Eq. (22), we find that

$$dx_1=\frac{\partial x_1}{\partial q_1}dq_1+\frac{\partial x_1}{\partial q_2}dq_2+\frac{\partial x_1}{\partial t}dt=q_2dq_2+g\cos(t)dt \tag{26}$$

and Eq. (20) gives

$$dx_2=\frac{\partial x_2}{\partial q_1}dq_1+\frac{\partial x_2}{\partial q_2}dq_2+\frac{\partial x_2}{\partial t}dt=dq_2 \tag{27}$$

The virtual displacements, which belong to the null space of the constraint matrix A (see in Eq. (4)), satisfy the relation (see Eq. (16))

$$\delta x_1-x_2\delta x_2=0 \tag{28}$$

in the coordinate system (x_1, x_2). This relation is automatically satisfied for arbitrary values of δq_2 because from Eqs. (26) and (27), we see that $\delta x_1=q_2\delta q_2$, and $\delta x_2=\delta q_2$ so that the left-hand side of Eq. (28) becomes $(q_2-q_2)\delta q_2$, which is identically zero for *arbitrary* values of δq_2. Hence, the equation of motion (24) for the coordinate q_2 represents a dynamical system that is unconstrained.

We have now come down to a more precise understanding of the conventional view, namely, that the 2-DOF holonomic system described by Eq. (18) has been "reduced" to an *unconstrained* single degree-of-freedom system given by Eq. (24), whose initial conditions can be arbitrarily prescribed! This is seen in the equation for the coordinate q_2 in Eq. (25), in which, as observed before, the parameters c_2 and c_4 can be arbitrarily prescribed.

We note that this "reduction" given by the single degree-of-freedom system (24) (recall here, $n=2$, $m=h=1$, and $n-h=1$) fundamentally involves the change in coordinates we undertook. For the new coordinate q_1, we used the expression φ_1 in Eq. (12), knowing that the constraint is $\varphi_1(t)\equiv 0$. As importantly, we hit upon a series of fortuitous happenings, which are: (i) we *chose* a coordinate $q_2=x_2$, which permitted $\det(J)$ to be nonzero not just locally, but globally, i.e., over the entire space of configuration coordinates (x_1, x_2), (ii) we knew the explicit equation of motion for q_2 from the second equation in Eq. (18), which we obtained from Eq. (5), because we chose $q_2=x_2$; this explicit equation is a very recent addition to our understanding of constrained motion, and (iii) as seen in Eq. (18), the second equation involves x_1, and, fortunately, by using the constraint $\varphi_1(t)=0$, we were able to express x_1 as a function of $q_2(x_2)$ and t, thereby obtaining the equation of motion for q_2—our "reduced" equation—in the form $\ddot{q}_2=f(q_2, \dot{q}_2, t)$, with arbitrary values for the initial conditions.

It should be kept in mind that the unconstrained 2-DOF system upon being constrained by the holonomic constraint $\varphi_1(t)=0$ continues to be a 2-DOF system. Our fortuitous transformation $(x_1, x_2)\rightarrow(q_1, q_2)$, as mentioned above, causes the equation of motion for $q_1(t)$ to be $q_1(t)\equiv 0$, which leads to the trivial equation of motion of the constrained system, $\ddot{q}_1=0$, along with the trivial initial conditions $q_1(0)=\dot{q}_1(0)=0$ that describes its evolution in time. We shall call $\ddot{q}_1=0$, along with these (zero) initial conditions, the "shadow" equation, to emphasize that, though trivial, it has a significant role in our understanding of how we got the equation of motion for the coordinate q_2, which we *chose* here to be x_2.

There are other interpretations/algorithms that have evolved over the years about this idea of "reduction" of the constrained 2-DOF system considered here to an unconstrained SDOF system, and, more generally, the "reduction" of an unconstrained n-DOF system subjected to h independent holonomic constraints to a "reduced" unconstrained $(n-h)$ degree-of-freedom subsystem describing the constrained motion of the n-DOF system, along, of course, with h shadow equations.

The first algorithm is to start with the Lagrangian of the 2-DOF unconstrained system in Eq. (11) which is

$$L=\frac{\dot{x}_1^2+\dot{x}_2^2}{2}-\frac{k_1x_1^2+k_2x_2^2}{2} \tag{29}$$

and, using the constraint equation $\varphi_1(t)=0$ that is given in Eq. (12), replace $x_1(t)$ in Eq. (29) by $(1/2)x_2^2(t)+g\sin(t)$, and similarly $\dot{x}_1$ using Eq. (13). The new expression (in the case of holonomic systems) continues to be the Lagrangian of the constrained system. It is

$$L=\frac{1}{2}[(x_2\dot{x}_2+g\cos(t))^2+\dot{x}_2^2]-\frac{1}{2}\left[k_1\left(\frac{1}{2}x_2^2+g\sin(t)\right)^2+k_2x_2^2\right] \tag{30}$$

Having eliminated the coordinate x_1 from Eq. (29), this approach now obtains the Lagrange equation of motion for the coordinate x_2 of the constrained system using the Lagrangian in Eq. (30). After some algebra, one finds that the equation of motion obtained is indeed the second equation in Eq. (25) (recall, we *chose* $q_2=x_2$). The initial conditions for this equation are those obtained directly from Eq. (11) for the coordinate x_2. While this does give the correct "reduced" equation of motion for the constrained system, it completely side-steps the following issues: (i) it assumes that "substitution" of x_1 in terms of x_2 and t, thereby using the constraint in the Lagrangian of the unconstrained 2-DOF system in Eq. (29), gives the correct Lagrangian for the coordinate x_2 of the constrained system, which is correct, (ii) it assumes such a substitution can always be done in an explicit manner, and (iii) it loses track of the fact that the unconstrained 2-DOF system in Eq. (11), when further subjected to the constraint, remains a 2-DOF constrained system, and (iv) it has no conceptual realization that along with the equation of motion for x_2, the other equation that completes the 2-DOF description of the constrained system is the "shadow" equation in the coordinate $q_1=\varphi_1$, namely the equation, $\ddot{q}_1=0$, with all the initial conditions for this differential equation set to zero.

We now take up each of the above-mentioned points. (i) The algorithmic approach described above of "substituting" the constraint into the Lagrangian of the unconstrained system to obtain the Lagrangian for the constrained system, leading to the "reduced" equation of the constrained system, has been so widely used for about 200 years, that the manner in which it is obtained has, it seems, eventually disappeared from our collective memory. Slowly, it gradually came to be a truth applicable to all constrained systems. It was only in 1949 that the mathematician/physicist Hamel discovered, through this algorithm's use in a specific example, that such a "substitution" of the constraint in the Lagrangian led him to the wrong equations of motion, and so was born the famous Hamel's Paradox, which remained unanswered by him and his contemporaries—who included individuals like Wyel, Lanczos, Gelfand, and Dirac—until very recently [19, 20]. (ii) Consider a constraint like $x_1^p+x_1=x_2^q+x_2^r$ where the integers $p\neq q\neq r$ and p, q, r are each greater than 5. Here, "substitution" of x_1 in terms of x_2 by using this constraint in the Lagrangian of the unconstrained 2-DOF system would be infeasible because the expression for x_1 as an explicit function of x_2 (and vice-versa) is not obtainable in terms of elementary functions and would, in general, be quite difficult to find. (iii) One loses sight of the fact that, as in Eq. (20), we *chose* q_2 to be x_2. We could just as well have chosen q_2 to be a wide list of entities, like, $1+x_2^3$, $1+e^{x_2}$, or $1+\tanh(x_2)$, which each make $\det(J)$ (globally) nonzero in any finite domain D_q. This could have made determining the reduced equation of motion $\ddot{q}_2=f(q_2, \dot{q}_2, t)$ substantially more difficult, in general.

A second approach would be to express the x_1 and x_2 in the Lagrangian (29) of the 2-DOF unconstrained system in terms of

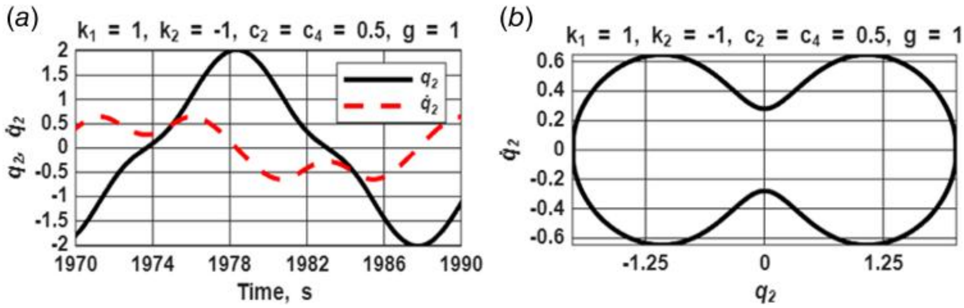

Fig. 1 Constrained nonautonomous system obtained from Eq. (24)

the new coordinates q_1 and q_2 using Eqs. (22) and (23) noting that $q_1 = 0$ and $q_2 = x_2$, so that the Lagrangian can be written as:

$$L(q_2, \dot{q}_2, t) = \frac{1}{2}[(q_2\dot{q}_2 + g\cos(t))^2 + \dot{q}_2^2] - \frac{1}{2}[k_1((1/2)q_2^2 + g\sin(t))^2 + k_2 q_2^2] \tag{31}$$

which is the same Lagrangian as in Eq. (30) with x_2 replaced by q_2. Lagrange's equation using this Lagrangian gives Eq. (24).

From Eq. (25), we observe that in the (q_1, q_2) coordinates, since $q_1(t) = \varphi(t) \equiv 0$ for all t in this 2-DOF system, we can side-step the computation of $q_1(t)$, using only the differential equation for $q_2(t)$, namely, the "reduced" equation (24). This has consequences when we compute the response of holonomically constrained 2-DOF systems using Eqs. (18) and (25). Though analytically these are equivalent ways of describing the constrained system through our one-to-one (global) mapping from $(x_1, x_2) \to (q_1, q_2)$, they can produce different results computationally.

To illustrate this, consider the constrained system with the parameters $k_1 = 1$, $k_2 = -1$, $g = 1$ and the initial conditions $c_2 = c_4 = 0.5$. Note that the 2-DOF unconstrained system, Eq. (11), is now unstable. Figure 1(*a*) shows the numerically computed response of the constrained system over the time interval [1970–1990] secs. when $q_2(t)$ is computed using the differential equation in Eq. (24). No computation is needed for q_1, since it is always zero. The computation (and all computations in this paper) is done using MATLAB's order (8–9) Runge–Kutta (RK) variable time-step integrator; the relative and absolute error tolerances used are 2.5×10^{-13} and 1.0×10^{-14}, respectively. Figure 1(*b*) shows the phase portrait of this nonautonomous system.

Having obtained $q_2(t) = x_2(t)$ and $\dot{q}_2(t) = \dot{x}_2(t)$ numerically, we obtain $x_1(t)$ and $\dot{x}_1(t)$ numerically by simply using Eqs. (22) and (23). Thus, in this case, the "reduced" equation obviates the need to use Eq. (18). For convenience, we denote $x_1(t)$ and $\dot{x}_1(t)$ so obtained numerically by $\tilde{x}_1$ and $\dot{\tilde{x}}_1$, respectively. We note that the "reduced" equation now provides a computationally valid solution in the coordinate system (x_1, x_2) because from it we have computationally determined $x_2(t) = q_2(t)$, $\dot{x}_2(t) = \dot{q}_2(t)$, $x_1(t) = \tilde{x}_1(t)$, and $\dot{x}_1(t) = \dot{\tilde{x}}_1(t)$, so that both the equations of constraint (22) and (23) are numerically satisfied; again, the tildas indicate that these quantities are obtained from the numerical solution of Eq. (24). Despite the computational inaccuracies introduced by the order (8–9) RK integrator used in computing the solution of Eq. (24), we shall assume it to be "accurate," and use $\tilde{x}_1(t)$ as a benchmark for further computations described below. Had we used the 2-DOF system (18) directly to numerically compute the response of the system in the (x_1, x_2) coordinate system, we would find that numerical errors creep into its computation and though theoretically $x_1(t)$ must analytically be the same as $\tilde{x}_1$, computationally, their differences grows over time. Figure 2(*a*) shows the error $\tilde{x}_1(t) - x_1(t)$, $x_1(t)$ having been computed using Eq. (18). One of the main reasons for this is the difficulty in maintaining the constraints during the integration. Figure 2(*b*) shows the error in the satisfaction of the constraint $\varphi_1(x_1(t), x_2(t), t) = 0$ given in Eq. (12). The reason for this error will be explained later (see the paragraphs before Eq. (129))

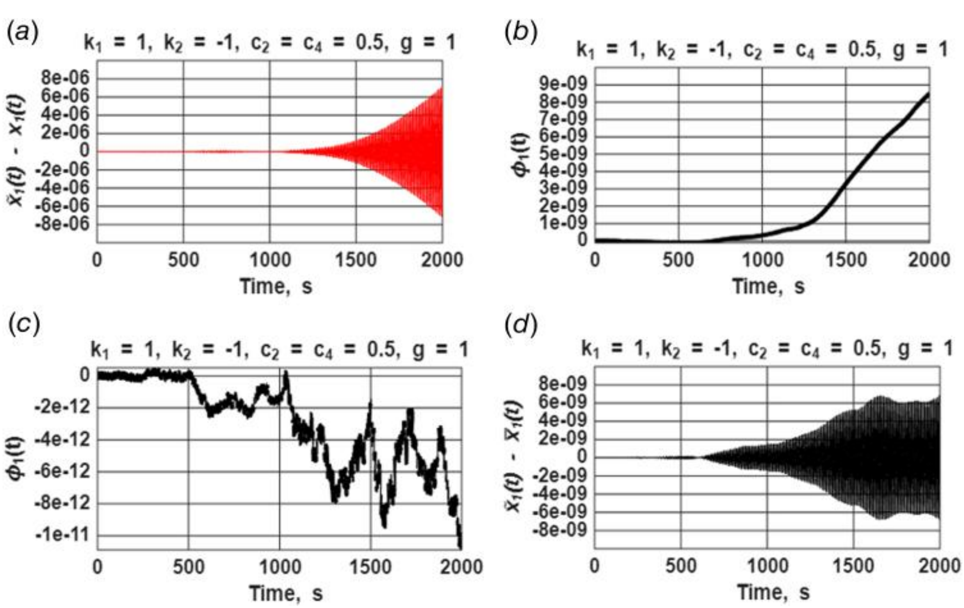

Fig. 2 Results of integration of Eq. (18)

As seen from Fig. 2(b), the error in satisfaction of the constraint increases with time and is several orders of magnitude greater than the error tolerances with which the integration of Eq. (18) is performed. The integration of Eq. (18) is performed using the four-element state vector $[x_1, x_2, \dot{x}_1, \dot{x}_2]^T$. A reduction in this error can be effected if in the RK integration scheme, when prescribing the four first-order differential equations to numerically compute the state vector from Eq. (18), $\frac{d\dot{x}_1}{dt}$ is prescribed, not as $\dot{x}_1$ (which is part of the state vector), but as $x_2\dot{x}_2 - g\cos(t)$, as required by Eq. (13) for the constrained system. The resulting error in the satisfaction of the constraint $\varphi_1(t)$ is shown in Fig. 2(c), a reduction of about three orders of magnitude. One can further improve the estimate of the component $x_1(t)$ (shown as $\bar{x}_1(t)$ in Fig. 2(d)), computing it by using Eq. (12) with $x_2(t)$ obtained from the improved four-element state vector computation. A comparison with Fig. 2(a) shows the improvement, again a reduction of three orders of magnitude. Computational methods to reduce the errors, such as these, in the integration of Eq. (18), are an area of current research activity, though the root reason for these computational errors appears to have gone unnoticed.

The direct brute-force numerical computation of the 2-DOF system using Eq. (18) can be made more accurate by observing that the constraint, $x_1(t) = (1/2)x_2(t)^2 + g\sin(t)$, allows the determination of $x_1(t)$ once $x_2(t)$ is computed. Hence one of the dependent variables, x_1, in Eq. (18) is dependent on the other, x_2. And the differential equation for x_2 is the lower equation in Eq. (18), whose right-hand side is a function of $(x_1, x_2, \dot{x}_2, t)$, which can be made a function of $(x_2, \dot{x}_2, t)$ using the constraint. The resultant differential equation would, of course, be just Eq. (24), since we chose q_2 to be x_2; and this differential equation can be numerically solved for x_2 and $\dot{x}_2$. We have obviated the need to compute $x_1(t)$ and $\dot{x}_1(t)$, done earlier in the brute-force method, by noting that they can be computed from $x_2(t)$ and $\dot{x}_2(t)$. This idea can be applied to systems with more than two degrees-of-freedom. We inspect the n-DOF equations of motion of the constrained system given by Eq. (5) and separate the dependent variable into two categories: those that are independent, and those that are dependent by virtue of the presence of the constraints. We then perform computations to determine the independent variables as far as possible; the time evolution of the dependent ones is obtained from these computations by using the constraints.

In summary, from a computational standpoint, higher accuracy in numerical computation is obtained by using the reduced equation for $q_2(t) = x_2(t)$ and side-stepping the computation of $q_1(t)$ since $q_1(t)$ is always zero. In this example x_1 and $\dot{x}_1$ can be explicitly obtained from q_1 and $\dot{q}_1$, and so the behavior of the system in the (x_1, x_2) coordinate system can be more reliably computed. When using the 2-DOF representation (18) of the constrained system with a four-element state vector, improved accuracy is delivered in this example by prescribing, in the RK computational scheme, $\dot{x}_1$ using the constraint (13). An improved estimate of $x_1(t)$ can be obtained by using the constraint $\varphi_1 = 0$, along with the computed value of $x_2(t)$ obtained in the four-element state vector in the RK computation.

Had we not obtained a one-to-one mapping that is globally valid, we would have to dissect the (x_1, x_2) plane and choose a different coordinate $q_2(x_1, x_2)$ for different D_x patches to ensure that the mapping is one-to-one. The behavior of the system would then need to be obtained in the different patches using these different coordinates, and the orbits appropriately "glued" together at the edges of these patches. While this can, of course, be conceptually done—recall, in each patch the reduced system is, in general, nonlinear, and therefore generally not amenable to analytical solution—it entails considerable computational effort along with concomitant computational inaccuracies, and, more importantly, it makes it more difficult to understand the (global) behavior of the system. Along with our choice of the coordinate q_1 in Eq. (19), had the coordinate q_2 not been chosen to be x_2—for example, we could have chosen $q_2 = x_1 - x_2 - (1/2)x_2^2$, which also provides a one-to-one global mapping—we would need to find the differential equation of motion governing this coordinate using Eqs. (18) and (13). Though we know theoretically that with the mapping $(x_1, x_2, t) \to (q_1, q_2, t) = (0, q_2, t)$ both x_1 and x_2 are functions of only q_2 and t, finding the differential equation for the time-evolution of the coordinate q_2 could now become a nontrivial endeavor. We illustrate some of these notions with the following two examples.

Example 2.

This example shows the change that occurs if the constraint in Example 1 is slightly modified to

$$\varphi_1(x_1, x_2, t) := x_1^2 + x_1 - \frac{1}{2}x_2^2 - x_2 - g\sin(t) = 0 \tag{32}$$

with g an arbitrarily given number, keeping the same unconstrained 2-DOF system as described previously in Eq. (11). This constraint yields the additional constraint

$$\tilde{\varphi}_1(x_1, x_2, t) := \dot{x}_1(2x_1 + 1) - \dot{x}_2(x_2 + 1) - g\cos(t) = 0 \tag{33}$$

that the dynamical system must also satisfy.

Upon differentiating the constraint (32) twice with respect to t, we get the constraint equation

$$\underbrace{[2x_1 + 1 \quad -(x_2 + 1)]}_{A}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} = \underbrace{\dot{x}_2^2 - 2\dot{x}_1^2 - g\sin(t)}_{b} \tag{34}$$

where we have shown the entities A and b preparatory to our use of Eq. (5). Noting that the column vector a, giving the acceleration of the unconstrained 2-DOF system (11) remains unchanged, we obtain $Aa = -2k_1x_1^2 + k_2x_2^2 - k_1x_1 + k_2x_2$. The equation of motion of the constrained 2-DOF system, using Eq. (5), becomes

$$\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix}-k_1x_1 \\ -k_2x_2\end{bmatrix} + \frac{1}{(2x_1+1)^2 + (x_2+1)^2}\begin{bmatrix}2x_1 + 1 \\ -(x_2+1)\end{bmatrix} \times \\ (\dot{x}_2^2 - 2\dot{x}_1^2 - g\sin(t) + 2k_1x_1^2 - k_2x_2^2 + k_1x_1 - k_2x_2) \tag{35}$$

We next consider the initial conditions for the constrained system (35) at time $t = 0$. The constants c_i, $i = 1, \ldots, 4$ that give the initial values of $x_i(0)$, $\dot{x}_i(0)$, $i = 1, 2$ in Eq. (11) can no longer be chosen arbitrarily since they need to satisfy the constraints (32) and (33). Solving Eq. (32) for x_2, we obtain

$$x_2 = -1 \pm \sqrt{1 + 2x_1^2 + 2x_1 - 2g\sin(t)} \tag{36}$$

Similarly from Eq. (33), by using Eq. (36) we have

$$\dot{x}_2 = \frac{\dot{x}_1(2x_1 + 1) - g\cos(t)}{x_2 + 1} = \pm\frac{\dot{x}_1(2x_1 + 1) - g\cos(t)}{\sqrt{1 + 2x_1^2 + 2x_1 - 2g\sin(t)}} \tag{37}$$

From Eqs. (36) and (37), we see that while $x_1(0) = c_1$ and $\dot{x}_1(0) = c_3$ can be chosen arbitrarily, once the values of c_1 and c_3 are chosen,

$$x_2(0) = c_2 = -1 \pm \sqrt{1 + 2c_1^2 + 2c_1} \tag{38}$$

$$\dot{x}_2(0) = c_4 = \pm\frac{c_3(2c_1 + 1) - g}{\sqrt{1 + 2c_1^2 + 2c_1}} \tag{39}$$

The initial conditions for the constrained system (35) are $x_1(0) = c_1$, $x_2(0) = c_2$, $\dot{x}_1(0) = c_3$, and $\dot{x}_2(0) = c_4$, in which c_1 and c_3 can have arbitrary values, and c_2 and c_4 are determined by Eqs. (38) and (39). The constants c_2 and c_4 depend on the sign chosen before the radicals on the right-hand sides in Eqs. (38) and (39), once the values of c_1 and c_3 are assigned.

We can rewrite the constraint $\varphi_1 = 0$ in the form $2(x_2 + 1)^2 - 4(x_1 + 1/2)^2 = 1 - 4g\sin(t)$, which is a hyperbola with asymptotes $x_2 + 1 = \pm\sqrt{2}(x_1 + 1/2)$, which intersect at $(-1/2, -1)$. When $1 - 4g\sin(t) = 0$, the hyperbola degenerates to a pair of straight lines with slope $\pm\sqrt{2}$ through this point of intersection, and the point $(-1/2, -1)$ satisfies the constraint, making Eq. (35) singular at that point.

Now consider, as in Example 1, a one-to-one mapping (coordinate change) from $(x_1, x_2) \rightarrow (q_1, q_2)$. Letting

$$q_1 = \varphi(x_1, x_2, t) \equiv 0 \tag{40}$$

the equation of motion for the coordinate $q_1(t)$ is simply $\ddot{q}_1(t) = 0$, along with the initial conditions $q_1(0) = \dot{q}_1(0) = 0$. The coordinate q_2 needs to be chosen so that the determinant of the Jacobian of the transformation is nonzero. Choosing

$$q_2 = 2x_1(x_2 + 1) + x_2 \tag{41}$$

we get

$$J = \begin{bmatrix} 2x_1 + 1 & -(x_2 + 1) \\ 2(x_2 + 1) & 2x_1 + 1 \end{bmatrix}$$

whose determinant is $(2x_1 + 1)^2 + 2(x_2 + 1)^2$, which is nonzero in any small domain D_x in the (x_1, x_2) plane that does not contain the point $(-1/2, -1)$, the point where Eq. (35) is singular. We know that except at this point, at which the one-to-one property of our mapping breaks down, for all other points, our mapping $(x_1, x_2, t) \rightarrow (q_1, q_2, t) = (0, 2x_1(x_2 + 1) + x_2, t)$ is one-to-one, so that x_1 and x_2 are both functions of only the coordinate q_2 and t. We note that at the point $(-1/2, -1)$, there is no coordinate $q_2(x_1, x_2, t)$ that can be chosen to make the mapping one to one. From Eq. (32) we observe that no trajectory of the constrained system can pass through this point when $|g| < 1/4$. We are now left with the task of finding the equation of motion for the coordinate q_2 in Eq. (41), in the form $\ddot{q}_2 = f(q_2, \dot{q}_2, t)$ from Eqs. (32), (33), and (35), which may not be an easy endeavor.

To obviate this difficulty, one could also choose $q_2 = x_1$, which would yield a one-to-one mapping except along the line $x_2 = -1$ in the (x_1, x_2) plane. The equation of motion for $q_2 = x_1$ can be obtained from the first equation in Eq. (35), by expressing x_2 and $\dot{x}_2$, on its right-hand side, as functions of x_1, $\dot{x}_1$, and t using Eqs. (36) and (37). When expressing x_2 as a function of x_1 and t, the sign before the radical in Eq. (36) must be chosen to be the same as that chosen in Eq. (38), and similarly, when expressing $\dot{x}_2$ the sign before the radical in Eq. (37) must be chosen the same as in Eq. (39). For a domain D_x in the vicinity of the line $x_2 = -1$ and $x_1 \neq -1/2$, one would need to switch to a different function, say, $q_2 = x_2$.

The alternative algorithmic approach to get the "reduced" equation of motion would be to solve the quadratic equation $x_1^2 + x_1 = \frac{1}{2}x_2^2 + x_2 + g\sin(t)$ for x_1 as a function of x_2 (or vice-versa) and t as in Eq. (36), which will give two roots, and substitute the roots in the Lagrangian of the unconstrained system in Eq. (29). This will yield *two* "reduced" equations of motion in the coordinate $x_2(x_1)$ depending on the sign of the quantity under the square-root in Eq. (36), with the initial conditions required to be chosen with the appropriate sign before the radical, as explained earlier. The algorithmic approach, which has no notion of coordinate changes, does not readily recognize that to cover the entire (x_1, x_2) plane (without the point $(-1/2, -1)$) one needs four reduced equations, two for $q_2 = x_1$ and two for $q_2 = x_2$.

One sees that though theoretically possible, the determination of the "reduced" equation in the coordinate q_2 given in Eq. (41) could get arguably difficult even with a low-dimensional 2-DOF system, and a simple constraint like Eq. (32).

Example 3.

Consider again the unconstrained 2-DOF system described by Eq. (11) on which the constraint

$$\frac{1}{p}x_1^p + x_1 = \frac{1}{s}x_2^s + \frac{1}{r}x_2^r,\ p \neq r \neq s,\ p, s, r \geq 6 \tag{42}$$

is imposed, where p, s, r are integers, with s and r being even.

Hence, we have the two constraints

$$\varphi(x_1, x_2) := \frac{1}{p}x_1^p + x_1 - \frac{1}{s}x_2^s - \frac{1}{r}x_2^r = 0 \tag{43}$$

$$\tilde{\varphi}(x_1, x_2) := x_1^{p-1}\dot{x}_1 + \dot{x}_1 - x_2^{s-1}\dot{x}_2 - x_2^{r-1}\dot{x}_2 = 0 \tag{44}$$

where the second is obtained by differentiating Eq. (43) with respect to t.

The explicit equation of motion of the constrained 2-DOF dynamical system is simple to obtain by using Eq. (5). Differentiating Eq. (44) with respect to t, we get the constraint equation

$$\underbrace{[\overbrace{(x_1^{p-1} + 1}^{\alpha},\ -\overbrace{(x_2^{s-1} + x_2^{r-1})}^{\beta}]}_{A}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$
$$= \underbrace{[(s-1)x_2^{s-2} + (r-1)x_2^{r-2}]\dot{x}_2^2 - (p-1)x_1^{p-2}\dot{x}_1^2}_{b}$$

where we have shown the entities A and b in the notation used in Eq. (5). Denoting $\alpha := x_1^{p-1} + 1$ and $\beta := (x_2^{s-1} + x_2^{r-1})$, the Moore–Penrose inverse of $A = [\alpha, \beta]$ is given by

$$A^+ = \frac{1}{(\alpha^2 + \beta^2)}\begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

and since (recall, $a = -[k_1x_1, k_2x_2]^T$ is the acceleration of the unconstrained system, as in the previous example)

$$Aa = -\alpha k_1 x_1 + \beta k_2 x_2$$

the equation of motion of the constrained 2-DOF system, using Eq. (5), is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -k_1x_1 \\ -k_2x_2 \end{bmatrix} + \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}\frac{\{[(s-1)x_2^{s-2} + (r-1)x_2^{r-2}]\dot{x}_2^2 - (p-1)x_1^{p-2}\dot{x}_1^2 + \alpha k_1x_1 - \beta k_2x_2\}}{(x_1^{p-1} + 1)^2 + (x_2^{s-1} + x_2^{r-1})^2}b \tag{45}$$

The initial conditions $c_i, i = 1, \ldots, 4$, of the unconstrained 2-DOF system described by Eq. (11) need to satisfy the constraints; they cannot be chosen arbitrarily. As mentioned before, from the arbitrarily chosen (real) numbers $c_2(=x_2(0))$ and $c_4(=\dot{x}_2(0))$, we need to find $c_1(=x_1(0))$ and $c_3(=\dot{x}_1(0))$ so that Eqs. (43) and (44) are satisfied.

We see that the denominator in the second member on the right-hand side of Eq. (45) is nonzero when p is odd. When $p \geq 6$ is even, the denominator is zero at the point $(-1, 0)$, which does not satisfy the constraint (42), and hence no trajectory of the constrained system goes through this point.

We now consider the determination of the "reduced" equation. Assuming, first, that $p \geq 6$ is an odd integer, with $p \neq r \neq s$. Consider now a change in coordinates (mapping) from $(x_1, x_2) \rightarrow (q_1, q_2)$ with the coordinate

$$q_1 = \frac{1}{p}x_1^p + x_1 - \frac{1}{s}x_2^s - \frac{1}{r}x_2^r \tag{46}$$

Since $q_1(t) \equiv 0\ \forall t$, the coordinate q_1 of the constrained system executes the trivial, shadow motion given by $\ddot{q}_1 = 0$, along with the initial conditions $q_1(0) = \dot{q}_1(0) = 0$. We now need to find a suitable independent coordinate $q_2(x_1, x_2)$ so that the determinant of the Jacobian $J = \dfrac{\partial(q_1, q_2)}{\partial(x_1, x_2)}$ is (locally) not zero. If we choose

$$q_2 = x_2 \tag{47}$$

then

$$J = \begin{bmatrix} x_1^{p-1} + 1 & -(x_2^{s-1} + x_2^{r-1}) \\ 0 & 1 \end{bmatrix}$$

and $\det(J) = 1 + x_1^{p-1} > 1$ globally, since x_1 is raised to an even power, and p is odd.

The lower equation in Eq. (45) gives the "reduced" equation of motion of the constrained system, which reads

$$\begin{aligned}(\alpha^2 + \beta^2)\ddot{q}_2 = &-k_2 q_2(\alpha^2 + \beta^2) - \beta[(s-1)q_2^{s-2} + (r-1)q_2^{r-2}]\dot{q}_2^2 \\ &+ \beta(p-1)x_1^{p-2}\dot{x}_1^2 - \alpha\beta k_1 x_1 + \beta^2 k_2 q_2 = -k_2\alpha^2 q_2 \\ &- \beta[(s-1)q_2^{s-2} + (r-1)q_2^{r-2}]\dot{q}_2^2 \\ &+ \beta[\underbrace{(p-1)x_1^{p-2}\dot{x}_1^2 - k_1(x_1^{p-1}+1)x_1}_{\gamma}]\end{aligned} \tag{48}$$

From Eq. (48), we see that to obtain the "reduced" equation for the coordinate q_2 defined in Eq. (47), we need to express γ and α in analytical form as a functions of x_2 and $\dot{x}_2$ by using Eqs. (43) and (44), which appears to be a difficult task, if at all possible.

When $p \geq 6$ is an even integer, with $p \neq r \neq s$, one could choose

$$q_2(x_1, x_2) = (x_1^{p-1} + 1)x_2 \tag{49}$$

so that

$$J = \begin{bmatrix} x_1^{p-1} + 1 & -(x_2^{s-1} + x_2^{r-1}) \\ (p-1)x_1^{p-2}x_2 & x_1^{p-1} + 1 \end{bmatrix} \tag{50}$$

whose determinant is $(x_1^{p-1}+1)^2 + (p-1)x_1^{p-2}(x_2^s + x_2^r)$. Since s and r are even, the determinant is zero at $(-1, 0)$, a point that does not satisfy the constraint (42). If the assumption that s and r are both even is relaxed, the determinant is zero on curves in the (x_1, x_2) plane, increasing the complexity of the mapping; we have made this assumption in our definition of the constraint to avoid this distraction.

We have thus obtained mappings $(x_1, x_2) \to (q_1, q_2) = (0, q_2)$ that are one-to-one when p is odd. When p is even, we have a one-to-one mapping except at one point in the (x_1, x_2) plane that does not satisfy the constraint. Theoretically, when p is even, this assures us that, away from the vicinity of this point, x_1 and x_2 are both functions of only q_2 given in Eq. (49). However, the determination of the equation of motion in the form $\ddot{q}_2 = f(q_2, \dot{q}_2, t)$ is problematic in this example, if not arguably impossible, despite having in hand the explicit equation of motion of the constrained system in the (x_1, x_2) coordinates given in Eq. (45). We are thus in a situation in which, from a practical point of view, the "reduced" equation appears to be unavailable, and it appears here that we are forced to consider the equation of motion of the constrained 2-DOF system in the original (x_1, x_2) coordinates. The algorithmic approach here appears doubtful, since x_1 and $\dot{x}_1$ cannot be easily expressed as functions of x_2 and $\dot{x}_2$ (and vice-versa) in (closed) analytical form using elementary functions, and hence, the "substitution" of x_1 and $\dot{x}_1$ in terms of x_2 and $\dot{x}_2$ in the Lagrangian (29) of the 2-DOF unconstrained system poses a significant problem when attempting to obtain the "reduced" equation of motion.

Thus, reducing this 2-DOF system with one holonomic constraint to a "reduced" system with one degree-of-freedom is difficult, if not arguably impossible.

The general ideas illustrated above for unconstrained 2-DOF systems that are holonomically constrained can be generalized to unconstrained n-DOF systems subjected to m independent holonomic constraints. One can then use the m constraints $\varphi_i(x, t) := \varphi_i(x_1, x_2, \ldots, x_n, t) = 0,\ i = 1, 2, \ldots, m,$ to form the first m independent Lagrange coordinates

$$q_i(t) = f_i(\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t)), \quad i = 1, 2, \ldots, m \tag{51}$$

where the smooth functions, f_i, are appropriately chosen. The remaining $r = (n-m)$ coordinates (subscript "r" for "reduced" coordinates)

$$q_r := (q_{m+1}(x), \ldots, q_n(x)) \tag{52}$$

must then be *chosen* so that the mapping $x := (x_1, x_2, \ldots, x_n) \to (q_1, q_2, \ldots, q_n) := q$ is (locally) one-to-one. (We note, in passing, that the general equation of constrained motion (5) is valid whether the constraints are independent or not. While independence is often easy to check for systems with holonomic constraints, it can become difficult when the constraints are nonholonomic.)

In choosing the variables in the reduced coordinate q_r one observes that: (1) such a mapping may not be valid globally (i.e., for all points in $x \in \mathbb{R}^n$), but only over smaller subdomains D_x in x-space, and (2) in each such subdomain, one must be able to obtain the "reduced" representation (in the chosen generalized coordinate q_r) of the system, i.e., obtain the equations of motion in the form $\tilde{M}(q_r, t)\ddot{q}_r = f(q_r, \dot{q}_r, t)$ where $\tilde{M}$ here is an $(n-m)$ by $(n-m)$ matrix and f is an $(n-m)$-dimensional column vector. As mentioned earlier, when a global one-to-one mapping cannot be found, a mapping will only be one-to-one in a certain subdomain. For each such subdomain D_x, a different set of Lagrange coordinates would need to be chosen to keep the mapping one-to-one, and so there will be different "reduced" representations of the constrained system for different subdomains. To get a picture of the global behavior of the constrained system one would then need to appropriately "glue" the behavior—obtained from these different reduced representations in the different subdomains—along the boundaries of the contiguous subdomains. Getting such a picture of the global behavior can often be difficult even for modest values of $(n-m) \sim 5$–10, since the reduced representations in the subdomains are usually nonlinear and, generally, cannot be solved analytically, forcing one to rely on computational methods, which have their own problems.

When considering an unconstrained n-DOF system subjected to m independent holonomic constraints, the determination of the $(n-m)$ "reduced" equations represents, in general, a far greater challenge when compared to the 2-DOF systems used in our examples for illustration. While theoretically true that an unconstrained n-DOF dynamical system on which m holonomic constraints are imposed can always be conceptually thought of as being locally "reduced" to an unconstrained $(n-m)$-DOF (unconstrained) system, obtaining the equations of motion for such a "reduced" system can be challenging in practice, the challenge becoming substantially more daunting when one is interested in a set of "reduced" $(n-m)$-DOF equations of motion that are *globally* valid, as pointed out before in the simple examples considered. The presence of such a globally valid "reduced" set of equations of motion is commonly, and perhaps somewhat glibly, assumed in the conventional notion of mechanics (e.g., Refs. [4,6]).

3 Nonholonomically Constrained Systems

We begin this section with two important features that, though known, seem to have gone sufficiently unrecognized in the conventional view of constrained motion. The first is that the principle of stationary action does not apply to nonholonomic systems, and the second is that the substitution of the constraint equation into the Lagrangian of the unconstrained system, as we did in obtaining the reduced equation in holonomic systems, leads to an expression that is not, in general, the Lagrangian for the constrained nonholonomic system.

We begin by providing a brief background on the first feature. Consider the orbit, O, of an n-DOF mechanical system from a configuration S_1 at time t_1 to a configuration S_2 at time t_2. We can visualize this orbit as the motion of a mobile moving in the n-dimensional configuration space described by the n Lagrange coordinates, $q_i(t)$, $i = 1, 2, \ldots, n$, for $t_1 \le t \le t_2$. The smooth motion of the mobile (orbit O) satisfies Lagrange's equations, and its coordinates take on values that correspond in configuration space to the point S_1 at time t_1 and to the point S_2 at time t_2, respectively. We assume that t_1 and t_2 are "close" to one another. We can now express Hamilton's principle in two forms.

(1) We consider the family of all possible geometrical paths in configuration space—call it family H—in this n-dimensional configuration space that leave point S_1 at time t_1 and reach S_2 at time t_2. The first form of Hamilton's principle affirms that: (i) the orbit O of the dynamical system, which satisfies Lagrange's equations, is a member of the family H, and (ii) $\int_{t_1}^{t_2} L dt$ has a stationary value for the orbit O of the system. Using the standard convention in variational calculus, the stationarity property can be written as

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{53}$$

where the δ here signifies variations of the integral when considering different paths in H close to the dynamical orbit O. The variations at the endpoints S_1 and S_2 are taken to be zero. This principle is often called by physicists, the principle of stationary action, with $\int_{t_1}^{t_2} L dt$ termed the action.

(2) Let at each instant of time t the orbit O pass through the point $q_i(t)$, $i = 1, 2, \ldots, n$. Consider a varied path $q_i(t) + \delta q_i(t)$, where $\delta q_i(t)$ is the ith component of the *virtual* displacement n-vector at time t. A virtual displacement at time t is defined as an n-vector $\delta q(t) = [\delta q_1(t), \ldots, \delta q_n(t)]^T$ such that $A(q, \dot{q}, t)\delta q(t) = 0$; it belongs to the null space of the matrix A, and it need not be infinitesimal. The varied path, like the orbit O, leaves the point S_1 at time t_1 and reaches the point S_2 at time t_2 so that $\delta q_i = 0$, $\quad i = 1, 2, \ldots, n$, at t_1 and t_2. The second form of Hamilton's principle states that

$$\int_{t_1}^{t_2} \delta L dt = 0 \tag{54}$$

These conditions, in each of the two versions are necessary and sufficient for having a dynamical orbit of the dynamical system. From a practical viewpoint, we use (the converse of) Hamilton's principle to obtain the equations of motion for the dynamical system, and with these equations, obtain the consequent motion using suitable initial conditions in both position and velocity at time t_1, instead of specifying the two positions that the orbit goes through at times t_1 and t_2.

For holonomic systems, the two versions of Hamilton's principle are equivalent [4]. For nonholonomic systems, the correct equations of motion are obtained from the second version of Hamilton's principle [4, 21, 22]; the first version is not applicable, and it leads to wrong equations of motion.

For a holonomic system, we can take the dimension n of the n-dimensional configuration space to equal the number of degrees-of-freedom, k, and for holonomic systems the two forms of Hamilton's principle are equivalent [4]. Every point in this $n(k)$-dimensional space is accessible from any other point in it by a suitably smooth geometric path. Since the number of degrees-of-freedom of the "reduced" dynamical system is also $n(k)$, the orbit O of the dynamical system resides in this $n(k)$-dimensional space. The search from the possible smooth geometric paths that make the action stationary yields the dynamical orbit O since the dynamical orbit, which satisfies Lagrange's equations, also belongs to this $n(k)$-dimensional space. However, for nonholonomic systems, this is not true.

To see why this is so, consider a mechanical system, described by n coordinates $q_i, i = 1, \ldots, n$, that is subjected to $m = h + u \le n$ constraints, h of them being independent holonomic and u being independent nonholonomic, the number of degrees-of-freedom of the system is $k = n - m$. In principle, the h holonomic constraints can be eliminated (locally) by coordinate transformations, as we did in the previous section. The remaining $n - h = n - (m - u) = (n - m) + u = k + u$ coordinates are needed for the specification of the system. Thus, the minimum number of coordinates needed to specify the configuration of the system in the presence of nonholonomic constraints is $w := k + u$, which exceeds the number of degrees-of-freedom of the system. Every point in this w-dimensional configuration space is accessible from any other point in it by a suitably smooth geometric path. However, since the number of degrees-of-freedom is $k < w$, the manifold that the dynamical system can access is only k-dimensional. Consider a point S_1 at time t_1 in the w-dimensional coordinate space as the starting point of our mobile; by assumption, it also belongs to the k-dimensional space accessible to the dynamical system. Consider an arbitrary point S_2 at time t_2 which belongs to this w-dimensional coordinate space with smooth geometric paths connecting the two points S_1 and S_2 in it. For starters, this point may not lie in the k-dimensional manifold accessible to the dynamical system from S_1. If we consider the family H of smooth geometric paths that start at S_1 at time t_1 and arrive at S_2 time t_2 in the w-dimensional coordinate space, there can be a member that makes the action stationary $\left(\delta \int_{t_1}^{t_2} L dt = 0\right)$, but that member will not necessarily be the orbit O of the dynamical system that starts from S_1 at time t_1 since the dynamical system cannot access the point S_2. And even if S_2 does belong to the k-dimensional space, which is accessible to the dynamical system, the actual orbit of the dynamical system would, in general, not be a member of the family H for which the action is stationary. That is, even if S_2 at time t_2 lies on a dynamical orbit O that also passes through the point S_1 at time t_1, the principle of stationary action does not, in general, give the correct equations of motion for nonholonomically constrained systems.

It is worth mentioning that physicists to date have focused mainly on holonomic constraints (see, e.g., Refs. [5–8]), and among many, the principle of stationary action is almost gospel. For example: "The chief law of physics, the pinnacle of the whole system is, in my opinion, the *principle of least action*, which involves the four universal coordinates in a perfectly symmetrical form.," and, "Among the more or less general laws, the discovery of which characterize the development of physical science during the last century, the principle of Least Action is at present certainly one which, by its form and comprehensiveness, may be said to have approached most closely to the ideal aim of theoretical inquiry. Its significance, properly understood, extends not only to mechanical processes, but also to thermal and electrodynamic problems." [23].

Though it has been theoretically established that the equations of motion for nonholonomic systems cannot be correctly obtained by the principle of stationary action, there are very few, if any, simple examples in the literature that illustrate this fact, seemingly causing it to have not yet emerged to the level of scientific attention it deserves today [4, 21]. And yet, there are numerous examples of nonholonomic systems both in nature and in engineered systems. In what follows we illustrate, by a simple example, the validity of the two features mentioned at the beginning of this section and uncover some new results in the process. Starting with familiar unconstrained systems, several simple examples of nonholonomic systems are then presented that show definite features of nonholonomic systems, several of which contradict the prevailing conventional view.

Example 4.

A. Consider an unconstrained system describing the motion of a particle in three dimensions that has no force acting on it and is described by the unconstrained equations of motion given by

$$\begin{aligned} &\ddot{x}=0,\ x(0)=a,\ \dot{x}(0)=p \\ &\ddot{y}=0,\ y(0)=b,\ \dot{y}(0)=q \\ &\ddot{z}=0,\ z(0)=c,\ \dot{z}(0)=r \end{aligned} \tag{55}$$

where $a, b, c, p, q,$ and r are (real) constants. The dynamics of the unconstrained (free) particle has three important features that we will need to use later on. (i) The acceleration of the particle in each direction is zero because it is not subjected to any impressed forces. (ii) The constants $a, b, c, p, q,$ and r can be arbitrarily chosen because the particle is unconstrained; every value of each of these constants is admissible. (iii) The unconstrained system has the trivial solution: $x(t)=pt+a,\ y(t)=qt+b,$ and $z(t)=rt+c$. The motion of the particle, for any (every) given set of initial conditions, is, as expected, always along a straight line with constant velocity, in three-dimensional configuration space.

We now impose on this system the nonholonomic constraint,

$$\psi(t):=\dot{x}z-\dot{y}=0 \tag{56}$$

Our focus is on how the orbits of the unconstrained system are altered by the imposition of this constraint, assuming that the initial conditions of the unconstrained system satisfy the constraint. We shall show that for some specific values of the parameters $p, q,$ and r, in Eq. (55), which are consistent with this constraint, the unconstrained system's straight-line orbit coincides with that of the corresponding constrained system that emerges after the constraint is imposed. In other words, there are special orbits of the *unconstrained* system that remain completely unaffected by the presence of the constraint. They are special because they demand special initial conditions. These special orbits of the unconstrained system remain unaffected because they (intrinsically) happen to satisfy the constraint $\psi(t)=0$ for all t. From the perspective of understanding the changes in the motion of the unconstrained system after the imposition of the constraint on it, such special orbits will be referred to as "trivial" orbits—orbits of the *unconstrained* system that remain unaffected by the presence of the constraint, i.e., the constraint imposes no change in them.

Differentiating this equation with respect to time, we get the constraint equation

$$\begin{bmatrix} z & -1 & 0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}=-\dot{x}\dot{z}$$

so that the correct equations of motion of the constrained system are given by

$$\begin{aligned} &\ddot{x}=-\dot{x}\dot{z}z/(1+z^2), && x(0)=a, && \dot{x}(0)=p \\ &\ddot{y}=\dot{x}\dot{z}/(1+z^2), && y(0)=b, && \dot{y}(0)=cp \\ &\ddot{z}=0, && z(0)=c, && \dot{z}(0)=r \end{aligned} \tag{57}$$

We will refer to this set of equations in Eq. (57) as $A1$.

Note that here we have one nonholonomic constraint ($m=u=1$), three coordinates ($n=3$), and no holonomic constraints ($h=0$). The constrained system has therefore two degrees-of-freedom, $k=n-m=2$. The number of coordinates, $w=k+u=3$, needed to describe the system exceeds the number of degrees-of-freedom, k, by the number of nonholonomic constraints, here 1. As mentioned before, the constant q in the initial condition describing the unconstrained system cannot now be chosen arbitrarily but must equal cp now, for consistency with the imposed constraint.

Observe that the system described by the first and third equations in Eq. (57) is "closed" since these two equations are all that is necessary to obtain $x(t)$ and $z(t)$, along with their derivatives, given the initial conditions. Once $x(t)$ and $z(t)$ are obtained, $y(t)$ and $\dot{y}(t)$ are then obtained from the second equation, by simple quadrature. Another way of saying this is that one does not need the second equation in Eq. (57) to obtain $y(t)$ and $\dot{y}(t)$; for $\dot{y}(t)$ can be found by using the constraint (56) once $\dot{x}(t)$ and $z(t)$ are known, and then $y(t)$ is found by simple quadrature. Thus, the essential equations are the first and third equations in Eq. (57), along with Eq. (56).

With the initial conditions shown in Eq. (57), the constrained dynamical system has the following solution:

When $r\neq 0$,

$$\begin{aligned} &\dot{x}(t)=\frac{p\sqrt{1+c^2}}{\sqrt{1+(rt+c)^2}}, && x(t)=\frac{p\sqrt{1+c^2}}{r}\left[\sinh^{-1}(rt+c)-\sinh^{-1}(c)\right]+a, \\ &\dot{y}(t)=\frac{p\sqrt{1+c^2}(rt+c)}{\sqrt{1+(rt+c)^2}}, && y(t)=\frac{p\sqrt{1+c^2}}{r}\left[\sqrt{1+(rt+c)^2}-\sqrt{1+c^2}\right]+b, \\ &\dot{z}(t)=r, && z(t)=rt+c \end{aligned} \tag{58}$$

When $r=0$,

$$x(t)=a+pt,\ y(t)=b+cpt,\quad \text{and } z(t)=c \tag{59}$$

as can be seen by taking the limit as $r\to 0$ of the expressions for $x(t)$, $y(t)$, and $z(t)$ in Eq. (58).

Upon denoting $rt+c:=\sinh(\alpha(t))$ and $pt=\beta(t)$, Eq. (58) can be rewritten as:

$$\begin{aligned} &x(t)=\frac{\beta(t)\cosh[\alpha(0)]}{\Delta}\{\alpha(t)-\alpha(0)\}+a \\ &y(t)=\frac{\beta(t)\cosh[\alpha(0)]}{\Delta}\{\cosh[\alpha(t)]-\cosh[\alpha(0)]\}+b \\ &z(t)=\sinh[\alpha(t)] \end{aligned} \tag{60}$$

where $\Delta=\sinh[\alpha(t)]-\sinh[\alpha(0)]$, and $\alpha(0)=\sinh^{-1}(c)$, $\beta(0)=0$.

When $\Delta=0$, then

$$y-cx=b-ca,\ z=c$$

Observe that the orbits O of the constrained system can be represented in the two-dimensional manifold described by the coordinates (α,β), showing that the number of degrees-of-freedom, k, of the system equals 2; however, the description of the system requires three coordinates, such as x, y, and z, in Eq. (57).

In order to understand the dynamical behavior of the constrained system in greater detail, we now focus our attention on two specific sets of orbits that start out with special sets of initial conditions.

(i) When $p=0$, Eq. (58) shows that the orbits of the constrained system are given by

$$x(t)=a,\ y(t)=b,\ z(t)=rt+c \tag{61}$$

so that the equations of motion of the constrained particle, and its initial conditions, become

$$\begin{aligned} &\ddot{x} = 0,\ x(0) = a,\ \dot{x}(0) = 0 \\ &\ddot{y} = 0,\ y(0) = b,\ \dot{y}(0) = 0 \\ &\ddot{z} = 0,\ z(0) = c,\ \dot{z}(0) = r \end{aligned} \tag{62}$$

Observe that (i) Eq. (61) shows that the motion is along a straight line in configuration space through the point (a, b, c) at time $t = 0$, with constant velocity r in the z-direction, just like the straight line, constant velocity, motions of the unconstrained particle in Eq. (55), and (ii) no additional force is required to be applied on the constrained particle to ensure that it satisfies the constraint, since its acceleration in Eq. (62) is zero, just as in Eq. (55) for the unconstrained motion of the free particle. This is because the orbits in Eq. (61) automatically satisfy the constraint (56), since along these orbits $\dot{x} = \dot{y} = 0$. In fact, the orbits in Eq. (61) are exactly the same as those of the *unconstrained* particle when its initial conditions are specifically chosen to be those in Eq. (62), namely,

$$(a, b, c, p, q, r) = (a, b, c, 0, 0, r) \tag{63}$$

With these special initial conditions, the differential equations of motion, Eq. (55), of the unconstrained system, become identical to those for the constrained system, Eq. (62); they both yield the orbits given in Eq. (61). Thus, the orbits of the constrained system are the same as those of the unconstrained system.

Orbits of the unconstrained system, such as these, which remain unchanged upon the imposition of constraints on the system, will be called trivial orbits.

(ii) When $r = 0$, the orbits of the constrained system are given by Eq. (59), which are:

$$x(t) = a + pt,\ y(t) = b + cpt, \quad \text{and } z(t) = c \tag{64}$$

from which we infer that the equations of motion of the constrained system are

$$\begin{aligned} &\ddot{x} = 0,\ x(0) = a,\ \dot{x}(0) = p \\ &\ddot{y} = 0,\ y(0) = b,\ \dot{y}(0) = cp \\ &\ddot{z} = 0,\ z(0) = c,\ \dot{z}(0) = 0 \end{aligned} \tag{65}$$

Going through a similar analysis as before, we see that these differential equations of motion of the constrained system are the same as those of the unconstrained system (55) when the parameters defining the initial conditions in Eq. (55) are chosen to be those in Eq. (65), namely,

$$(a, b, c, p, q, r) = (a, b, c, p, cp, 0) \tag{66}$$

As mentioned earlier, the constrained particle therefore has the same orbits as the unconstrained particle. These orbits are also trivial orbits. From Eqs. (62) and (65) we see that the special case when $p = r = 0$ leads to the trajectory $x(t) = a$, $y(t) = b$, and $z(t) = c$, which then becomes an equilibrium position of the dynamical system, since $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$.

We have thus illustrated, by example, an important result. There can be orbits of the unconstrained system, which, for special sets of values of the initial conditions, undergo no change upon the imposition of constraints on the unconstrained system, since these orbits intrinsically satisfy the constraints for all time t. Here we have shown that when $p = 0$, or when $r = 0$, the orbits of the unconstrained system remain unaffected by the imposition of the constraint. Such trivial orbits can also arise for systems with holonomic constraints.

Suppose now that we have two points, S_1 at time $t_1 = 0$, and another point S_2 at time $t_2 > 0$, through which the orbit O given by Eq. (60) passes. This orbit, which starts at point S_1 and arrives at S_2 at time t_2 must lie entirely in the two-dimensional space with coordinates (α, β). Thus, starting from the point S_1, the system can *only* access those points S_2 that lie in this two-dimensional space.

Recalling that we need three coordinates to describe the system, consider the first form of Hamilton's principle. For this principle to yield the correct orbit O, both S_1 and S_2 through which the orbit passes at times $t_1 = 0$ and t_2, respectively, must lie in the three-dimensional coordinate space ($w = 3$). Numerous smooth geometrical curves connect any two points in this three-dimensional coordinate space. If the point S_2 does not lie in the two-dimensional space dynamically accessible from S_1, then clearly $\delta\int_{t_1}^{t_2} L dt = 0$, in which the variation is carried out in the three-dimensional coordinate space, will not yield the correct orbit O. Let us then assume that S_2 (at time t_2) lies in this two-dimensional space, and that by setting $\delta\int_{t_1}^{t_2} L dt = 0$, we obtain a trajectory T that connects the points S_1 at time $t_1 = 0$ and S_2 at some time $t_2 > 0$. We now show that this trajectory T, obtained by using the calculus of variations in the three-dimensional coordinate space, is not, in general, the dynamical orbit O of the system.

Using the Lagrange multiplier $\mu(t)$, the augmented Lagrangian of the system when employing the principle of stationary action is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \mu(t)(\dot{x}z - \dot{y}) \tag{67}$$

For $\delta\int_{t_1}^{t_2} L dt = 0$, variational calculus yields the following set of equations (which we will refer to as $B1$) for the trajectory passing between the points S_1 and S_2 at the respective times $t_1 = 0$ and t_2.

$$\begin{aligned} &\ddot{x} = \dot{\mu}z + \mu\dot{z} = \frac{d(\mu z)}{dt}, && x(0) = a, && \dot{x}(0) = p \\ &\ddot{y} = -\dot{\mu}, && y(0) = b, && \dot{y}(0) = cp \\ &\ddot{z} = -\mu\dot{x} && z(0) = c, && \dot{z}(0) = r \end{aligned} \tag{68}$$

To these equations we add the equation of constraint

$$\dot{x}z = \dot{y} \tag{69}$$

so that we have four differential equations from which we may determine the four functions $\mu(t)$, $x(t)$, $y(t)$, and $z(t)$.

The first two equations in Eq. (68) can be integrated, so that $\dot{x} = \mu z + c_1$ and $\dot{y} = -\mu + c_2$, where c_1 and c_2 are arbitrary constants. Using (69) yields

$$\mu(t) = \frac{c_2 - c_1 z}{1 + z^2} \tag{70}$$

so that

$$\dot{x}(t) = \frac{c_1 + c_2 z}{1 + z^2} \text{ and } \dot{y}(t) = \frac{(c_1 + c_2 z)z}{1 + z^2} \tag{71}$$

Taking into account the initial conditions and using either of the equations in Eq. (71), we obtain

$$c_1 = p(1 + c^2) - cc_2 \tag{72}$$

where c_2 (or c_1) is *arbitrary*. Using the relations obtained in Eqs. (70)–(72), the third equation in Eq. (68) can then be rewritten as:

$$\ddot{z}(t) = \frac{c_1 c_2 z^2 + (c_1^2 - c_2^2)z - c_1 c_2}{(1 + z^2)^2} \tag{73}$$

Noting Eq. (72), we see that the solutions to Eq. (73) that are obtained from $B1$ form a one-parameter family $z = z(t, c_2)$ in which c_2 is any arbitrary (real) number; also, from Eq. (71), we see that $x = x(t, c_2)$ and $y = y(t, c_2)$. We thus conclude that trajectories T of the dynamical system $B1$ are not unique, and depend on the parameter c_2. (These are orbits of the dynamical system $B1$ in three-dimensional coordinate space, but we call them trajectories T, to help distinguish them, in our mind, from the correct orbits O given by $A1$.) In analytical dynamics, we expect the trajectories (orbits) of the dynamical system to be unique; the presence of nonuniqueness, as evidenced by the presence of a one-parameter family of solutions, illustrates that the use of the stationary action principle is, indeed, inappropriate when nonholonomic constraints are imposed on a dynamical system.

We can further investigate when the equation sets $B1$ and $A1$ give identical solutions, i.e., when the use of the principle of stationary action yields the same equations of motion as the correct equations of motion of the constrained system given in (57). Recall that now both dynamical systems have the same initial conditions, which are compatible with the constraint.

We begin by assuming that for some value of c_2, $B1$ and $A1$ *do* give identical solutions. We note that $A1$ gives

$$z(t) = rt + c \text{ and } \ddot{z}(t) = 0$$

Under our assumption, the set of equations in $B1$ must also have the solution $z(t) = rt + c$ along with $\ddot{z}(t) = 0$. In that case, the expression on the right-hand side of Eq. (73) must reduce to zero when $z(t) = rt + c$. Thus, our assumption requires that

$$c_1 c_2 (rt + c)^2 + (c_1^2 - c_2^2)(rt + c) - c_1 c_2 = 0$$

Using Eq. (72), this equation, which is a polynomial in t, reduces, after some algebra, to

$$a_1 t^2 + a_2 t + a_3 = 0 \tag{74}$$

where

$$\begin{aligned} a_1 &= c_2 r^2 [p(1 + c^2) - cc_2] \\ a_2 &= r(1 + c^2)[p^2(1 + c^2) - c_2^2] \\ a_3 &= p(1 + c^2)(pc - c_2). \end{aligned} \tag{75}$$

Thus, for the two sets $B1$ and $A1$ to give the same solutions for a common, consistent set of initial conditions, it is necessary and sufficient that each of the coefficients of the polynomial equal zero. When $p, r \neq 0$, from Eq. (75), we find that for a_3 to equal zero, we require $c_2 = pc$; this makes $a_1 = cp^2 r^2$, and $a_2 = rp^2(1 + c^2)$, which are not both zero, for all values of c. Hence, our assumption that $A1$ and $B1$ give identical solutions for some value of c_2 is incorrect when $p, r \neq 0$.

When $p = 0$, $r \neq 0$, for a_2 to equal zero in Eq. (75), we require $c_2 = 0$. Then since Eq. (72) gives $c_1 = -cc_2$ when $p = 0$, we find that $c_1 = 0$, and Eq. (70) gives $\mu(t) = 0$. From Eq. (67), we see that when the Lagrange multiplier $\mu(t)$ is identically zero; there is no constraint on the system (see Eq. (67)), and the differential equations consequently obtained by making the action stationary are $\ddot{x} = \ddot{y} = \ddot{z} = 0$.

Recall, that when $p = 0$ (see Eq. (61), and the discussion that follows) we have trivial orbits, which already satisfy the constraints and these orbits of the unconstrained system are unaffected by the presence of the constraint; they are identical to those of the constrained system for any given consistent set of special initial conditions. The stationary action principle, being applicable to an unconstrained system, thus gives the correct orbits of the constrained system, because the constrained system, in this case, has the same orbits as the unconstrained system.

Similarly when $r = 0, p \neq 0$, we know from Eq. (59) (see Eq. (64) and the discussion under item (ii)) that we again get trivial orbits. For the coefficients in Eq. (75) to equal zero, we require $c_2 = cp$. Since the trajectory T has $z = c$, $\mu(t) = \dfrac{c_2 - c_1 c}{1 + c^2} = c_2 - pc = 0$, again pointing out that the stationary action principle "sees" no constraint, since the trajectories automatically satisfy the constraint, and it yields the differential equations $\ddot{x} = \ddot{y} = \ddot{z} = 0$. The orbits of the constrained system coincide with the orbits of the unconstrained system, and, again, the principle of least action, being applicable to the unconstrained system, gives the correct orbit of the constrained system, for the special set of initial conditions that yield these trivial orbits.

We thus conclude that the principle of stationary action yields trajectories T that provide the correct orbits of the constrained dynamical system for the so-called trivial orbits—orbits of the *unconstrained* system that remain totally unaffected by the imposition of constraints on the unconstrained system. The reason we get the correct orbits is that for these trivial orbits, the orbits of the constrained system are identical to those of the unconstrained system, for which we know that the principle of stationary action is applicable.

In summary, in this example we have shown that the principle of stationary action gives trajectories T that are: (i) not unique since they belong to a one-parameter family of solutions of Eq. (73) in which the parameter c_2 can take arbitrary values, something quite unexpected (and incorrect) when dealing with the motion of a mechanical system, whose orbit for a given set of initial conditions is considered to be unique in analytical dynamics; and (ii) the principle of stationary action provides trajectories T, which are orbits O of the constrained system for "trivial" orbits. These are orbits of the unconstrained system that are unaffected by the presence of the constraint because these orbits intrinsically satisfy the constraint for all time t, and the principle of stationary action is applicable to unconstrained systems. Item (i) is sufficient to show that the equations obtained by making the action stationary cannot, in general, correctly describe the motion of the constrained system.

As pointed out in Eq. (60) through every point S_1 with coordinates (a, b, c) at time $t_1 = 0$ in the three-dimensional coordinate space, there passes a two-dimensional manifold, spanned by the different values that $\dot{x} = p$ and $\dot{z} = r$ can take, at the point (a, b, c). This two-dimensional manifold is generated by the different values p and r taken by $\dot{x}$ and $\dot{z}$ at (a, b, c) at time $t_1 = 0$; once c and p are fixed, the value of $\dot{y}$ at (a, b, c) is pc, which is then fixed. We can explicitly determine this two-dimensional manifold, Σ, in the configuration space of the system, because from Eq. (58), we have

$$\begin{array}{ll} (x - a)(\sqrt{1 + z^2} - \sqrt{1 + c^2}) - (y - b)(\sinh^{-1}(z) - \sinh^{-1}(c)) = 0, & r \neq 0 \\ y - cx = b - ca, \; z = c, & r = 0 \end{array} \tag{76}$$

since $z = rt + c$. The second equation follows from Eq. (59). The two-dimensional manifold Σ accessible to the system from the point (a, b, c) is given by Eq. (76). As seen, for an orbit that starts from a different point (a', b', c') in configuration space we will have a different two-dimensional manifold (surface) Σ' accessible to it.

Figure 3 shows two views of the two-dimensional (twisted) manifold Σ in configuration space that is accessible to an orbit that goes through the origin, i.e., through the point $(a, b, c) = (0, 0, 0)$, which is marked by an "X." Regions of the manifold (surface) where $z < 0$ are shown in blue, and regions where $z > 0$ are shown in cyan. The red traces show two representative orbits of the constrained dynamical system that go through the origin and lie in the surface Σ.

This figure allows us to visualize what happens if we choose two points in coordinate space, say, S_1 being the origin (marked by the black "X") where the trajectory starts at $t_1 = 0$, and an arbitrarily chosen point S_2, say, the point $(-4, -5, -6)$, shown by the black dot, where the trajectory ends at some time $t_2 = t* > 0$. These two points, as seen from the figure, allow numerous geometrically possible paths that join them. We may then find a trajectory

T that goes through these two points at times $t_1 = 0$ and $t_2 = t*$, respectively, by enforcing stationarity of the action between the points S_1 and S_2. However, the point S_2 does not lie on Σ, the two-dimensional manifold accessible to the dynamical system from the origin, and the trajectory T, if found, is not a dynamical orbit O of the system passing through the origin. As seen in the figure, no orbit of the constrained system that starts from the origin reaches the point S_2. Assume, further, that we choose, by some fluke, a point S_2 that does lie on Σ. What we have shown before analytically is that a trajectory T joining these two points by using stationarity of the action would again be incorrect when $r, p \neq 0$; it would not be a dynamical orbit O of the system. If $r = 0$, or, $p = 0$, we have trivial orbits O, and the principle of stationary action yields the correct orbits of the constrained system in this case. This is because these are orbits of the unconstrained system that are identical to those of the constrained system when the two systems have the same special initial conditions; and since the principle of stationary action gives the correct orbits for an unconstrained system, it also gives the correct orbits for the constrained system, in these cases. These trivial orbits, which lie in the surface Σ, and which are orbits of the unconstrained system that are unaffected by the presence of the constraint, are shown by the two black straight lines in the figure.

We now move on to illustrate the second feature mentioned at the beginning of this section. Consider the Lagrangian of the unconstrained system, which is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

This unconstrained system is next subjected to the constraint $\dot{y} = \dot{x}z$. Substituting this constraint in the Lagrangian as we did in Eq. (31) for the holonomic constraint in the previous section, the altered Lagrangian L^* is

$$L^* = \frac{1}{2}(\dot{x}^2 + \dot{x}^2 z^2 + \dot{z}^2) \tag{77}$$

This Lagrangian, which includes the constraint substituted in it, yields the "closed" set of Lagrange equations

$$\begin{aligned} \ddot{x} &= -2\dot{x}z\dot{z}/(1 + z^2) \\ \ddot{z} &= \dot{x}^2 z \end{aligned} \tag{78}$$

The last equation is incorrect. This illustrates that substitution of a nonholonomic constraint in the the correct Lagrangian of an unconstrained system gives a set of Lagrange equations of motion that are, in general, incorrect.

However in this case, we *can* find a Lagrangian for the constrained system, which gives the correct Lagrange equations obtained in Eq. (57). We begin by observing that by using our knowledge of the first and third equations of motion of the constrained system in Eq. (57), the Lagrange equations using L^* can be written as the "closed" system of equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}} - \frac{\partial L^*}{\partial x} = z\dot{x}\dot{z}, \quad \text{and} \quad \frac{d}{dt}\frac{\partial L^*}{\partial \dot{z}} - \frac{\partial L^*}{\partial z} = -z\dot{x}^2 \tag{79}$$

Observe that the right-hand sides in Eq. (79) are nonzero, showing again that the function L^* is not the correct Lagrangian for the constrained system.

We consider now a change in the independent variable $t \to \tau$, defined by $d\tau = N(z)dt$, where the coordinate function $N(z)$ is subject to determination so that in the variables x and z the equations of motion have the Lagrange form [24]. Here, z is obtained from the solution of Eq. (79). On making this coordinate transformation, we find that the specific choice

$$N(z) = 1/\sqrt{1 + z^2}$$

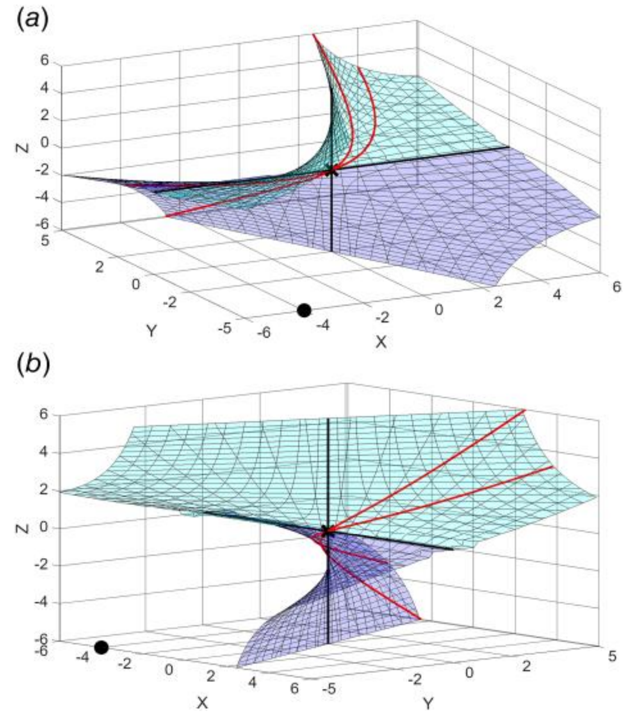

Fig. 3 Two views of the twisted two-dimensional surface Σ passing through the origin (marked by the black X). Dynamical trajectories, which satisfy Lagrange's equation, that start from the origin cannot pass through any arbitrary point in configuration space, like the one shown by the black dot at (−4, −5, −6).

leads to the Lagrangian

$$\hat{L} = \frac{1}{2}\left[x'^2 + \frac{1}{1 + z^2}z'^2\right] \tag{80}$$

where the primes denote differentiations with respect to τ. In terms of the new "time" coordinate τ, the Lagrangian $\hat{L}$ gives the Lagrange equations of motion,

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial x'}\right) - \left(\frac{\partial \hat{L}}{\partial x'}\right) = 0, \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial z'}\right) - \left(\frac{\partial \hat{L}}{\partial z'}\right) = 0$$

which are equivalent to the first and third equations in Eq. (57) that correctly describe the motion of the constrained dynamical system. That this is true for the specific choice of the function $N(z)$, is proved in Appendix B, since it involves some algebra. Thus, we have established that, like the unconstrained system, the constrained system, in this example, can also be described by a Lagrangian, though not L^*. However, the usefulness of $\hat{L}$ appears somewhat limited because in this case, it is ultimately obtained (using Eq. (79)) only after the equations of motion of the constrained system given in Eq. (57) are already known.

Lastly, we point out that the energy, E, of the dynamical system before and after the imposition of the constraint remains the same, for any set of compatible initial conditions that satisfy the constraint. The energy of the unconstrained system, now with the initial conditions in Eq. (57), is

$$E(t) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}(p^2 + (cp)^2 + r^2)$$

and it remains unchanged after the imposition of the constraints, as seen from Eqs. (58) and (59).

B. There is another circumstance when the stationary action principle yields the correct trajectories of motion for a nonholonomically constrained system. If a nonholonomically constrained system has an invariant set (manifold) such that on it the nonholonomic constraint becomes holonomic, then the principle of stationary action will correctly provide orbits of the constrained system that start on the manifold, since the stationary action principle is valid for holonomically constrained systems. We illustrate this with the following example.

Consider again the unconstrained free particle (with no impressed force on it) described by Eq. (55) that is subjected now to the constraint

$$\psi := (2x_1 + x_3^2)\dot{x}_1 - \dot{x}_2 = 0 \tag{81}$$

The equations of motion of the constrained system using Eq. (5) are:

$$\begin{aligned}
\ddot{x}_1 &= -\frac{2}{\Delta}(2x_1 + x_3^2)(\dot{x}_1 + x_3\dot{x}_3)\dot{x}_1, \quad && x_1(0) = a, \quad && \dot{x}_1(0) = p\\
\ddot{x}_2 &= \frac{2}{\Delta}(\dot{x}_1 + x_3\dot{x}_3)\dot{x}_1, && x_2(0) = b, && \dot{x}_2(0) = p(2a + c^2)\\
\ddot{x}_3 &= 0, && x_3(0) = c, && \dot{x}_3(0) = r
\end{aligned} \tag{82}$$

where $\Delta = 1 + (2x_1 + x_3^2)^2$, and only the initial values (a, b, c, p, r) are now arbitrary. Recall that the unconstrained system (55) has the initial values (a, b, c, p, q, r) that are all arbitrary.

Our knowledge of the constrained equation of motion, Eq. (82), shows that there is the invariant plane $x_3 = c, \dot{x}_3 = 0$, and on this plane the constraint (81) reduces to $(2x_1 + c^2)\dot{x}_1 - \dot{x}_2 = 0$, which is integrable $(x_1^2 + c^2 x_1 - x_2 = \text{constant})$ and holonomic! Hence, the principle of stationary action will then yield the correct equations of motion for orbits that start with the special initial conditions

$$(a, b, c, p, q, r) = (a, b, c, p, p(2a + c^2), 0)$$

since the principle is applicable to systems with holonomic constraints. Of course, to know that there is such an invariant manifold of a nonholonomic system, which changes the constraint to a holonomic one, one has to know the correct equations of motion of the constrained system, which, of course, is what the principle of stationary action is attempting to find in the first place!

As before, we have, in addition, trivial orbits, which are

$$x_1(t) = a,\ x_2(t) = b,\ x_3(t) = c + rt \tag{83}$$

that start, for the constrained system, from the initial conditions $(a, b, c, r) = (a, b, c, 0, r)$. Along these orbits we see, from the three right-hand sides of the equations in Eq. (82), that the acceleration equals 0. This shows that no additional force is needed to be applied to the unconstrained system, upon the imposition of the constraint if the system starts from these initial conditions; the constraints are automatically satisfied. Corresponding to these initial conditions of the constrained system we have the special initial conditions $(a, b, c, p, q, r) = (a, b, c, 0, 0, r)$ of the unconstrained system, which also generates the orbits given in Eq. (83). Since the trajectories of the unconstrained and constrained systems are identical for these trivial orbits, as before, the principle of stationary action, which, of course, holds for any unconstrained system, yields the correct orbits of the constrained system.

Example 5.

Consider a particle that moves in three-dimensional space and is subjected to a potential force, along with linear damping, and is described by

$$\begin{aligned}
\ddot{x} &= -V_x(x, y, z) - \delta_x\dot{x}, \quad && x(0) = a, \quad && x(0) = p,\\
\ddot{y} &= -V_y(x, y, z) - \delta_y\dot{y}, && y(0) = b, && y(0) = q,\\
\ddot{z} &= -V_z(x, y, z) - \delta_z\dot{z}, && z(0) = c, && z(0) = r
\end{aligned} \tag{84}$$

where V_x, V_y, and V_z denote partial derivatives of $V(x, y, z)$ with respect to x, y, and z, respectively. The parameters a, b, c, p, q, r, δ_x, δ_y, and δ_z are real constants.

We impose on this unconstrained dynamical system the nonholonomic constraint given by

$$\psi := \dot{y} - \alpha f(z)\dot{x} = 0 \tag{85}$$

where $\alpha \neq 0$ is a constant, and $f(z)$ is C^2.

The equation of motion of the constrained system is obtained by first differentiating Eq. (85) with respect to time to get the constraint equation

$$[-\alpha f(z),\ 1,\ \ 0]\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \alpha f'(z)\dot{x}\dot{z}$$

where the prime denotes differentiation with respect to z. Using Eq. (5) with $a = [-\bar{V}_x,\ -\bar{V}_y,\ -\bar{V}_z]^T$, where $\bar{V}_x := V_x + \delta_x\dot{x}$, $\bar{V}_y := V_y + \delta_y\dot{y}$, and $\bar{V}_z := V_z + \delta_z\dot{z}$, we get the following equation of motion of the constrained dynamical system:

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix}-(\bar{V}_x + \alpha f(z)\bar{V}_y + \alpha^2 f(z)f'(z)\dot{x}\dot{z})\\ -\alpha(\bar{V}_x f(z) + \alpha f(z)^2\bar{V}_y - f'(z)\dot{x}\dot{z})\\ -\bar{V}_z\Delta\end{bmatrix}, \quad \begin{bmatrix}x(0) = a,\ \dot{x}(0) = p\\ y(0) = b,\ \dot{y}(0) = \alpha p f(c)\\ z(0) = c,\ z(0) = r\end{bmatrix} \tag{86}$$

where $\Delta = 1 + \alpha^2[f(z)]^2$.

We consider here the case when $\delta_x, \delta_y, \delta_z = 0$, so that the unconstrained system is conservative. (The nonconservative case with damping is considered in the following section.) Thus, $\bar{V}_x = V_x$, $\bar{V}_y = V_y$, $\bar{V}_z = V_z$, in Eq. (86), and the Lagrangian for the conservative unconstrained system is then

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \tag{87}$$

When the constraint (85) is substituted in this expression, it becomes

$$L^* = \frac{1}{2}(\dot{x}^2 + \alpha^2 f(z)^2\dot{x}^2 + \dot{z}^2) - V(x, y, z) \tag{88}$$

which is *not*, as before, the Lagrangian for the constrained system, because it gives the Lagrange equation of motion

$$\ddot{z} = \alpha^2 f(z)f'(z)\dot{x}^2 - V_z$$

which is incorrect, in view of Eq. (86).

However, the energy is conserved in the following sense. Consider the total energy of the unconstrained system,

$$E(t) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z) \tag{89}$$

which this constraint conserves. When this unconstrained system is subjected to the initial conditions in Eq. (86), $E(0) = \frac{1}{2}(p^2 + \alpha^2 p^2 f(c)^2 + r^2) + V(a, b, c)$.

The derivative of $E(t)$ is then

$$\frac{dE}{dt} = \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} + \dot{x}V_x + \dot{y}V_y + \dot{z}V_x \quad (90)$$

and we find, after some algebra, that $\frac{dE}{dt} = 0$, upon using the expressions for $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$ from Eq. (86) in Eq. (90). This shows that the energy $E(t)$ in Eq. (89) is conserved before and after the imposition of the constraint, making it an invariant. These conclusions corroborate our results in the previous example, in which $V \equiv 0$.

Particularizing the potential energy of the conservative system to

$$V(x, y, z) = \frac{1}{2}(a_x x^2 + a_y y^2 + a_z z^2)$$

and taking $f(z) = \sin(z)$, the equations of motion of the (conservative) constrained system, Eq. (86), become

$$\begin{aligned}
&\ddot{x} = -[a_x x + \alpha \sin(z)(\alpha \cos(z)\dot{x}\dot{z} + a_y y)]/\Delta, && x(0) = a, \ \dot{x}(0) = p\\
&\ddot{y} = -\alpha[a_x x \sin(z) + \alpha a_y y \sin(z)^2 - \cos(z)\dot{x}\dot{z}]/\Delta, && y(0) = b, \ \dot{y}(0) = \alpha pc\\
&\ddot{z} = -a_z z, && z(0) = c, \ \dot{z}(0) = r
\end{aligned} \quad (91)$$

with $\Delta = 1 + \alpha^2 \sin^2(z)$.

The energy $E(t)$ of the system is conserved in the sense stated earlier.

Example 6.

A. Consider the unconstrained linear dynamical system, which is described by

$$\begin{aligned}
&\ddot{x} = -\alpha y, && x(0) = a, && \dot{x}(0) = p\\
&\ddot{y} = \alpha x, && y(0) = b, && \dot{y}(0) = q\\
&\ddot{z} = -1, && z(0) = c, && \dot{z}(0) = r
\end{aligned} \quad (92)$$

where $\alpha \neq 0$ is a (real) constant, and which is then subjected to the nonholonomic constraint

$$\psi := \dot{y} - z\dot{x} - \dot{z} = 0 \quad (93)$$

The first two equations in Eq. (92) form a "closed" system with a circulatory force, and the last equation is decoupled from this closed system. Furthermore, the unconstrained dynamical system has no equilibrium positions.

Using Eq. (5), the equation of motion of the constrained system is

$$\begin{aligned}
&\ddot{x} = (z - 2\alpha y + \alpha x z - z\dot{x}\dot{z})/(2 + z^2), && x(0) = a, \ \dot{x}(0) = p\\
&\ddot{y} = (\alpha x - 1 + \alpha x z^2 - \alpha y z + \dot{x}\dot{z})/(2 + z^2), && y(0) = b, \ \dot{y}(0) = cp + r\\
&\ddot{z} = (\alpha x - 1 + \alpha y z - z^2 - \dot{x}\dot{z})/(2 + z^2), && z(0) = c, \ \dot{z}(0) = r
\end{aligned} \quad (94)$$

The equilibrium positions require $\dot{x} = \dot{y} = \dot{z} = 0$ and are therefore solutions of the equations: $z - 2\alpha y + \alpha x z = 0$, $\alpha x - 1 + \alpha x z^2 - \alpha y z = 0$, $\alpha x - 1 + \alpha y z - z^2 = 0$. On substituting for αy from the former into the two latter equations, we obtain $(z^2 + 2)(\alpha x - 1) = 0$ so that the set of equilibrium positions is given by $x = 1/\alpha$, $z = \alpha y$, which is an infinite line of nonisolated equilibrium points.

B. Assume that we have a 3-DOF unconstrained system, like the one in Eq. (92) above, which is subjected to the holonomic constraint $\varphi_1 := z - x^2 = 0$. We now add a nonholonomic constraint, so that the unconstrained system has the set of constraints

$$\varphi_1 := z - x^2 = 0, \quad \text{and } \psi_1 := \dot{y} - z\dot{x} - \dot{z} = 0 \quad (95)$$

one of which is holonomic, and the other nonholonomic.

Observe that using $z = x^2$ in the second, nonholonomic constraint yields $\dot{y} - x^2\dot{x} - 2x\dot{x} = \frac{d}{dt}(y - x^3/3 - x^2) = 0$, so that we now actually have a set of two *holonomic* constraints:

$$y - x^3/3 - x^2 - d = 0, \quad \text{and } z - x^2 = 0 \quad (96)$$

where d is a (real) constant. The constant d is determined from the initial conditions.

For example, using the initial conditions shown in Eq. (92), we see that the constant d for the constrained system is obtained from knowledge of a and b. From Eq. (96), when $x(0) = a$ and $y(0) = b$, $d = b - a^3/3 - a^2$, and $z(0) = a^2$. Also, with $\dot{x}(0) = p$, from Eq. (96) $\dot{z}(0) = 2ap$, so that initial conditions for the constrained system with the constraints (95) are then given by

$$x(0) = a,\ y(0) = b,\ z(0) = a^2,\ \dot{x}(0) = p,\ \dot{y}(0) = a^2 p + 2ap,$$
$$\dot{z}(0) = 2ap$$

where a, b, and p are arbitrary (real) constants.

We see that the presence of the second nonholonomic constraint in the set of constraints (95) does not necessarily make the constrained system nonholonomic.

C. Consider a 3-DOF unconstrained system, like the one in Eq. (92), subjected to the set of two nonholonomic constraints

$$\psi_1 := \dot{y} - y\dot{x} - \dot{z} = 0, \quad \text{and } \psi_2 := y\dot{x} - \dot{z} = 0$$

These constraints reduce to a set of two holonomic constraints. Substituting for $\dot{z}$ from the second relation in the first gives $\dot{y} = 2y\dot{x}$. On changing coordinates to $\bar{x} = x$ and $\bar{y} = y\exp(-2x)$, this differential equation becomes $\dot{\bar{y}} = 0$ or $\bar{y} = d_1$, where d_1 is a constant. Hence, the general integral of $\dot{y} = 2y\dot{x}$ is $y = d_1\exp(2x)$. Using this expression for y in the second constraint, we get $d_1\exp(2x)\dot{x} = \dot{z}$, which can be integrated, to give $d_1\exp(2x) = 2z + d_2$, where d_2 is a constant. Noting that $y = d_1\exp(2x)$, this constraint reduces to $y - 2z = d_2$. Thus, the set of nonholonomic constraints are equivalent to the set of holonomic constraint

$$\varphi_1 := y - d_1 e^{2x} = 0 \text{ and } \varphi_2 := y - 2z - d_2 = 0\left(\text{or,} \quad \varphi_2 := z - \frac{d_1}{2}\exp(2x) + \frac{d_2}{2} = 0\right) \quad (97)$$

The constants d_1 and d_2 are obtained by using the initial conditions since the initial conditions need to satisfy the constraints at time $t = 0$. Note that the holonomic constraints (97) imply $\tilde{\varphi}_1 := \dot{y} - 2d_1\exp(2x)\dot{x} = 0$ and $\tilde{\varphi}_2 := \dot{y} - 2\dot{z} = 0$, which are constraints on the velocities.

For example, for the values of the initial conditions in Example 6A (Eq. (92)), d_1 and d_2 are determined as follows. When $b \neq 0$ in the constrained system, $d_1 = \exp(-2a)b$ and $d_2 = b - 2c$, and we obtain the following initial conditions for the constrained system:

$$x(0) = a,\ y(0) = b \neq 0,\ z(0) = c,\ \dot{x}(0) = \frac{r}{b},\ \dot{y} = 2r,\ \dot{z}(0) = r$$

where a, b, c, and r are arbitrary. When $b = 0$ in the constrained system, then $d_1 = 0$, $d_2 = -2c$, so that the initial conditions for the constrained system are:

$$x(0) = a,\ y(0) = 0,\ z(0) = c = -d_2/2,\ \dot{x}(0) = p,\ \dot{y} = 0,\ \dot{z}(0) = 0$$

where a, c, and p are arbitrary.

More generally, when we have a set of Pfaffian constraints, then their expressions (e.g., ψ_1 and ψ_2) are one-forms, and Frobeneus's theorem gives a necessary and sufficient condition for such a set of (nonholonomic) constraints to be integrable. However, when the set of constraints is nonlinear in the velocities, the theorem is not applicable. We next consider a set of constraints containing such nonholonomic constraints to which Frobeneus's theorem does not apply and show that they can also be reduced to a holonomic set.

D. Consider the set of constraints,

$$\psi_1 := \dot{x} - \dot{y}\dot{z} - \dot{x}\dot{z} = 0,\ \psi_2 := \dot{y} + \dot{y}\dot{z} + \dot{x}\dot{z} = 0 \quad (98)$$

Each constraint is nonholonomic and nonlinear in the velocities. Summing the two constraints, we get $\dot{x} = -\dot{y}$, and substituting for $\dot{x}$ in ψ_2 gives $\dot{y} = 0$, so that $\dot{x} = 0$. Hence, we obtain the two holonomic constraints $x = c$ and $y = d$, where c and d are constants.

Thus, we see that a set of constraints, in which each of the constraints is nonholonomic and nonlinear in the velocities, might be reducible to a set of holonomic constraints; that is, even if each constraint by itself is nonholonomic in a set of constraints, the set might still be holonomic.

Example 7.

We consider next a linear nonconservative mechanical system described by

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} 7 & 14 & 8 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, \quad \begin{matrix} x(0) = a, x(0) = p \\ y(0) = b, y(0) = q \\ z(0) = c, z(0) = r \end{matrix} \tag{99}$$

where a, b, c, p, q, and r, are arbitrary numbers. The matrix multiplying the column vector $[x, y, x]^T$ can be thought of as the stiffness matrix of the linear system. Its eigenvalues are {1,2,4}, and the equilibrium state $x = y = z = \dot{x} = \dot{y} = \dot{z} = 0$ of the system is stable. On this unconstrained system we now impose the nonholonomic constraint

$$\psi := \dot{x} - y\dot{z} = 0 \tag{100}$$

Differentiating the constraint equation with respect to time yields the constraint equation

$$[1 \quad 0 \quad -y] \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \dot{y}\dot{z}$$

and the equation of motion of the constrained system, using Eq. (5), is

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} (y^2 - 14y^3 - 7xy^2 - 8y^2 z + \dot{y}\dot{z})/(1 + y^2) \\ x \\ (y - 14y^2 - 7xy - 8yz - y\dot{y}\dot{z})/(1 + y^2) \end{bmatrix}, \quad \begin{matrix} x(0) = a, & \dot{x}(0) = br \\ y(0) = b, & \dot{y}(0) = q \\ z(0) = c, & \dot{z}(0) = r \end{matrix} \tag{101}$$

We shall use this example in the following section on stability, because it has some special features.

4 Stability of Constrained Systems

We begin by noting that, strictly speaking, the term stability of a system is adequate only for linear systems since all its solutions have the same stability properties. In nonlinear systems, some motions (e.g., equilibria) may be stable and others unstable. In this section we shall be concerned with the stability of such multiple equilibria.

Perhaps the most important issue faced in most previous studies on the stability of nonlinear dynamical systems subjected to nonholonomic constraints was the lack of knowledge of the actual equations of motion of these constrained systems. Without knowledge of the explicit equations of motion of such constrained systems, the determination of their equilibrium states (fixed points) is difficult to obtain, and investigations of their stability, even more so. This is perhaps why most stability studies of constrained systems in the past were targeted at specific, relatively simple, mechanical, and structural systems. Often, the Lagrange multipliers were obtained and the equations of motion for the constrained systems were thereby determined. However, the determination of these multipliers for even modest-sized nonlinear systems that have 5 to 10 degrees-of-freedom with three to four nonlinear nonholonomic constraints was generally quite difficult. Complex mechanical systems have numerous degrees-of-freedom and many constraints. This often made the Lagrange multiplier approach impractical. Previous investigators often used physical arguments to infer the equilibrium states (positions) of such constrained systems, and most often simply investigated whether the known equilibrium states of the unconstrained system continued to maintain their stability upon imposition of constraints. The recent development presented in Eq. (5), which provides explicit equations of motion for general nonlinear systems subject to holonomic and/or nonholonomic constraints, enables a systematic determination of their equilibrium states and facilitates rigorous stability analysis. As will be seen, the presence of constraints can engender additional equilibrium positions beyond those of the unconstrained system; also, they can eliminate equilibria that the unconstrained system has, and add new ones. More importantly, constrained motion often gives rise to nonisolated equilibrium states whose stability is more difficult to examine for lack of currently available analytical tools [24]. Yet, there is a considerable amount of literature on the stability of nonholonomic systems, though, as mentioned, most studies limit themselves to ascertaining whether the equilibrium points of the unconstrained system change stability upon the imposition of constraints, while some try to ascertain stability by trying to expand the solution of the constrained system around a set of equilibrium conditions. The determination of additional equilibrium points, different from those of the unconstrained system, appears to be obtained by most previous investigators either based on physical grounds pertinent to their understanding of the dynamics of a specific system under consideration, or intuition, or experience. As we shall see, even nonholonomic constraints can engender nonisolated (a continuum of) equilibrium states (manifolds).

The problem of handling these nonisolated equilibria has a long history going back to the work of Neimark and Fufaev [9], Whittaker [3], Bottema [25], Aizerman and Gantmacher [26], Rumyantsev [27], Karapetyan [28, 29], and many others. Yet significant gaps in our understanding of constrained systems still persist. For instance, the interesting problem of the "inversion" of the Lagrange–Dirichlet theorem for nonholonomic systems has been treated in many papers but has not been solved completely [30–33]. It is noteworthy that while one can obtain conditions related to when a nonholonomically constrained system becomes unstable, ascertaining the regions in phase (or parameter) space where the system is stable has often been far more challenging. Moreover, the prevalent literature affirms that nonholonomic systems cannot have isolated equilibrium states; they must have a continuum of equilibrium states [24], which Example 9A contradicts. There is also the conventional belief that the addition of a nonholonomic constraint to a holonomic system makes the dynamical system nonholonomic, and that the addition of a nonholonomic constraint to a nonholonomic system keeps it nonholonomic. Both these prevalent beliefs are shown to be false in Example 6B, 6C, and 6D.

In this section, we present some simple systems, many of them introduced in previous sections of the paper, and point out some remarkable features that have not been reported in previous studies. While our primary focus is on nonholonomic systems, we demonstrate through examples that holonomic systems can also exhibit unexpected and interesting constrained dynamics.

Before setting out on these examples, we prove an analytical result for determining the stability of both isolated and nonisolated equilibrium states of constrained systems. Our approach is based on the Lyapunov–Malkin theory [34, 35], which deals with cases where the characteristic equations of linearized systems have zero roots. We show the applicability of this result in providing more definitive answers regarding the stability of constrained dynamical systems through the examples that follow. Despite this, we find that our understanding of the stability of nonholonomic systems is still incomplete, as illustrated in Example 11.

Consider an unconstrained n-DOF system described in the introductory section earlier in which the matrix M and the n-vector Q do not depend explicitly on time t. We describe the unconstrained system using the n-vectors q and $\dot{q}$. We will assume that:

(a) The unconstrained system is subjected to the m independent consistent constraints (i.e., Rank$(A) = m$) described by

$$\varphi(q) = 0,\ \varphi = [\varphi_1, \ldots, \varphi_h]^T,$$
$$\psi(q, \dot{q}) = 0,\ \psi = [\psi_1, \ldots, \psi_u],\ u = m - h$$

in which $\psi(q, \dot{q}) = 0$ is such that

$$\psi(q, \dot{q})|_{\dot{q}=0} = 0 \tag{102}$$

and

(b) the constrained system allows an equilibrium state, which we shall take as $q = \dot{q} = 0$, with no loss of generality.

We perform a local coordinate change $q = q(\xi, \eta)$, $\xi \in \mathfrak{R}^h$, $\eta \in \mathfrak{R}^{n-h}$ such that $q(0, 0) = 0$ and $\varphi(q(\xi, \eta)) = \xi$. In the new coordinates, ξ and η, the constraints have the following forms

$$\xi = 0, \text{ and } \Psi(\eta, \dot{\eta}) = 0,\ \Psi(\eta, \dot{\eta})|_{\dot{\eta}=0} = 0 \text{ with Rank}\left(\frac{\partial \Psi}{\partial \dot{\eta}}\right) = u$$

(Note the difference between the previous ψ's in (a) and Ψ above.) Next, we partition the $(n-h)$-vectors η and $\dot{\eta}$ as $[x \quad y]^T$ and $[\dot{x} \quad \dot{y}]^T$, respectively, in the following way. Since Rank$\left(\frac{\partial \Psi}{\partial \dot{\eta}}\right) = u$, the implicit function theorem informs us that in the local vicinity of the point $\eta = \dot{\eta} = 0$, the equations $\psi(\eta, \dot{\eta}) = 0$ can be expressed uniquely as follows:

$$\dot{y} = f(x, \dot{x}, y),\ y \in \mathfrak{R}^u,\ x \in \mathfrak{R}^{n-m} \tag{103}$$

Thus, in the vicinity of the equilibrium state the constraint Ψ can be written as $\Psi(x, \dot{x}, y, \dot{y}) = \dot{y} - f(x, \dot{x}, y) = 0$. This last equality, upon using the assumption in (a), namely, that $\Psi(x, \dot{x} = 0, y, \dot{y} = 0) = 0$, gives $f(x, \dot{x} = 0, y) = 0$, which implies that

$$\left.\frac{\partial f}{\partial x}\right|_{(x,0,y)} = 0,\ \left.\frac{\partial f}{\partial y}\right|_{(x,0,y)} = 0 \tag{104}$$

From here on, for convenience, we shall denote by the subscript "0" the point $(x, \dot{x}, y) = (0, 0, 0)$, so that the subscript "0" on any expression means the evaluation of that expression at this point (0,0,0).

One can also assume that

$$\left(\frac{\partial \dot{y}}{\partial \dot{x}}\right)_0 = \left(\frac{\partial f(x, \dot{x}, y)}{\partial \dot{x}}\right)_0 = 0 \tag{105}$$

For if this is not so, then a simple change in coordinates to

$$\bar{x} = x \text{ and } \bar{y} = y - \left(\frac{\partial f(x, \dot{x}, y)}{\partial \dot{x}}\right)_0 \bar{x} \tag{106}$$

will ensure that $\left(\frac{\partial \dot{\bar{y}}}{\partial \dot{\bar{x}}}\right)_0 = 0$. This is because by differentiating the last equality in Eq. (106), and noting that $\bar{x} = x$, we get

$$\dot{\bar{y}} = \dot{y} - \left(\frac{\partial f}{\partial \dot{x}}\right)_0 \dot{\bar{x}} = f(x, \dot{x}, y) - \left(\frac{\partial f(x, \dot{x}, y)}{\partial \dot{x}}\right)_0 \dot{x}$$

where we have used Eq. (103) in the second equality. This leads to

$$\frac{\partial \dot{\bar{y}}}{\partial \dot{\bar{x}}} = \frac{\partial f}{\partial \dot{x}} - \left(\frac{\partial f}{\partial \dot{x}}\right)_0$$

from which it follows that $\left(\frac{\partial \dot{\bar{y}}}{\partial \dot{\bar{x}}}\right)_0 = 0$.

From Eqs. (104) and (105) we observe that the function $f(x, \dot{x}, y)$ has *no* linear terms in its Taylor series expansion in x, $\dot{x}$, and y about the point $(x, \dot{x}, y) = (0, 0, 0)$. Thus, $f(x, \dot{x}, y)$ represents purely the higher-order nonlinear terms at this point.

Using Eq. (5), the equation of motion in the vicinity of the equilibrium state $(\xi, \dot{\xi}, x, \dot{x}, y, \dot{y}) = (0, 0, 0, 0, 0, 0)$ has the explicit form (recall: $\xi, \dot{\xi} \in \mathfrak{R}^h$, $x, \dot{x} \in \mathfrak{R}^{n-m}$ and, $h + u = m$)

$$\begin{aligned} \ddot{\xi} &= 0 \\ \ddot{x} &= X(x, \dot{x}, y, \dot{y}), \\ \ddot{y} &= Y(x, \dot{x}, y, \dot{y}), \end{aligned} \tag{107}$$

for which the initial conditions are $\xi(0) = \dot{\xi}(0) = 0$, $x(0) = x_0$, $y(0) = y_0$, $\dot{x}(0) = \dot{x}_0$, $\dot{y}(0) = f(x_0, \dot{x}_0, y_0)$. At the equilibrium state, the accelerations are zero, and, therefore, thereat $X = 0$ and $Y = 0$. The stability of this equilibrium state is then the conditional stability of the trivial solution of Eq. (107).

We note that the trivial solution of Eq. (107) is conditionally stable relative to the set $C := \{(\xi, x, y, \dot{\xi}, \dot{x}, \dot{y}) : \xi = 0, \dot{\xi} = 0, \dot{y} = f(x, \dot{x}, y)\}$ if for any $\varepsilon > 0$, no matter how small, there exists a $\delta(\varepsilon) > 0$ such that from $(\xi(0), x(0), y(0), \dot{\xi}(0), \dot{x}(0), \dot{y}(0)) \in \{(\xi, x, y, \dot{\xi}, \dot{x}, \dot{y}) : \|(\xi, x, y, \dot{\xi}, \dot{x}, \dot{y})\| < \delta\} \cap C$ it follows that $\|(\xi(t), x(t), y(t), \dot{\xi}(t), \dot{x}(t), \dot{y}(t))\| < \varepsilon$ for every $t \geq 0$. But we know that the dynamics evolve on the manifold C, and consequently, the conditional stability of Eq. (107) is equivalent to the unconditional stability of the trivial solution of the dynamical system

$$\begin{aligned} \ddot{x} &= \tilde{X}(x, \dot{x}, y) \\ \dot{y} &= f(x, \dot{x}, y) \end{aligned} \tag{108}$$

where $\tilde{X}(x, \dot{x}, y) = X(x, \dot{x}, y, \dot{y} = f(x, \dot{x}, y))$. The relation between X and $\tilde{X}$ reveals that

$$\left(\frac{\partial \tilde{X}}{\partial x}\right)_0 = \left(\frac{\partial X}{\partial x}\right)_0 + \left(\frac{\partial X}{\partial \dot{y}}\frac{\partial f}{\partial y}\right)_0 = \left(\frac{\partial X}{\partial x}\right)_0 \tag{109}$$

The second member on the right-hand side drops out in view of Eq. (104). Similarly, from Eqs. (104) and (105)

$$\left(\frac{\partial \tilde{X}}{\partial y}\right)_0 = \left(\frac{\partial X}{\partial y}\right)_0, \quad \text{and} \quad \left(\frac{\partial \tilde{X}}{\partial \dot{x}}\right)_0 = \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \tag{110}$$

We now consider the linearization of Eq. (107) about the equilibrium state to get

$$\ddot{\xi} = 0$$

$$\ddot{x} = \left(\frac{\partial X}{\partial x}\right)_0 x + \left(\frac{\partial X}{\partial y}\right)_0 y + \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \dot{x} + \left(\frac{\partial X}{\partial \dot{y}}\right)_0 \dot{y} \tag{111}$$

$$\ddot{y} = 0$$

The last equality follows from the fact that the Taylor series expansion of $\dot{y} = f(x, \dot{x}, y)$ has no linear terms in x, $\dot{x}$, and y, as mentioned earlier.

Similarly, noting Eqs. (109) and (110) linearization of Eq. (108) about $(x, \dot{x}, y) = (0, 0, 0)$ (recall: in the neighborhood of the origin, at this point $\dot{y}$ is also zero, see Eqs. (104) and (105)) results in

$$\ddot{x} = \left(\frac{\partial X}{\partial x}\right)_0 x + \left(\frac{\partial X}{\partial y}\right)_0 y + \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \dot{x} \qquad (112)$$
$$\dot{y} = 0$$

The characteristic equation of the linear system (111) is

$$\Delta_1(\lambda) = \lambda^{2m} \det\left[\lambda^2 I_{n-m} - \lambda\left(\frac{\partial X}{\partial \dot{x}}\right)_0 - \left(\frac{\partial X}{\partial x}\right)_0\right] = 0 \qquad (113)$$

while that of the linear system (112) is

$$\Delta_2(\lambda) = \lambda^{u} \det\left[\lambda^2 I_{n-m} - \lambda\left(\frac{\partial X}{\partial \dot{x}}\right)_0 - \left(\frac{\partial X}{\partial x}\right)_0\right] = 0 \qquad (114)$$

Hence, $\Delta_1(\lambda) = \lambda^{m+h}\Delta_2(\lambda)$. We observe that all the eigenvalues of Eq. (112) are also the eigenvalues of Eq. (111). The characteristic equation of system (111) has at least $2m$ zero roots, and that of system (112) has at least u zero roots. *This is true regardless of whether the equilibrium state is isolated or not.*

Remark 1. We note that the linear system (112) is Lyapunov stable (the system (111) is conditionally stable with respect to the hyperplane $\{(\xi, x, y, \dot{\xi}, \dot{x}, \dot{y}):\xi = 0, \dot{\xi} = 0, \dot{y} = 0\}$) if and only if the system has no eigenvalues with positive real parts and the eigenvalues with vanishing real parts are semisimple (i.e., its algebraic multiplicity equals its geometric multiplicity). It should be noted that when the system has $2(n-m)$ nonzero eigenvalues, then the u-fold zero eigenvalue is semisimple. However, the nonlinear system (108) (the system (107)) is not necessarily stable (conditionally stable).

If at least one of the roots of Eq. (114) (or Eq. (113)) has a positive real part, then according to the Lyapunov theorem on instability in the first approximation, the trivial solution of system (108) is unstable [36], and consequently, the equilibrium state (0, 0, 0, 0, 0, 0) of system (107) is conditionally unstable. Note that if the equilibrium state of system (107) is conditionally unstable, then it is also unstable (without restrictions on the initial perturbations), but the reverse is not true. For example, when $h \geq 1$ the trivial solution of Eq. (107) is unstable since these equations have solutions for which $\xi(t) = \xi_0 + \dot{\xi}_0 t$ with arbitrarily small constants ξ_0 and $\dot{\xi}_0$, yet it can be conditionally stable (see Example 8 below). However, when Eq. (112) (or Eq. (111)) has no eigenvalue with a positive real part, the determination of stability/instability cannot usually be resolved easily, because there are eigenvalues that are equal to zero, i.e., the Lyapunov critical case occurs [36]. We next take this up.

In the neighborhood of the origin, i.e., $(x, \dot{x}, y) = (0, 0, 0)$, the form that Eq. (108) takes is

$$\ddot{x} = \left(\frac{\partial X}{\partial x}\right)_0 x + \left(\frac{\partial X}{\partial y}\right)_0 y + \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \dot{x} + \tilde{X}^*(x, \dot{x}, y) \qquad (115)$$
$$\dot{y} = f(x, \dot{x}, y)$$

where $\tilde{X}^*$ and f represent nonlinear terms. Since $f(x, 0, y) = 0$, the set of equilibria is given by

$$\left\{(x, \dot{x}, y):\dot{x} = 0, \left(\frac{\partial X}{\partial x}\right)_0 x + \left(\frac{\partial X}{\partial y}\right)_0 y + \tilde{X}^*(x, 0, y) = 0\right\} \qquad (116)$$

Suppose that the $(n-m)$ by $(n-m)$ matrix $\left(\frac{\partial X}{\partial x}\right)_0$ is nonsingular. Then, by the Implicit Function Theorem, in a sufficiently small neighborhood of the origin, there exists a unique function $x = \theta(y)$, which satisfies the second equation in Eq. (116), and in that case, the equilibrium point belongs to the u-dimensional manifold

$$\{(x, \dot{x}, y):\dot{x} = 0, x = \theta(y)\} \qquad (117)$$

showing that in this case, the dimension of the equilibrium manifold is equal to the number of velocity-dependent constraints, since $y \in \mathfrak{R}^u$. The implicit function theorem, of course, ensures that the function $x = \theta(y)$ satisfies the second equation in Eq. (116), so that

$$\left(\frac{\partial X}{\partial x}\right)_0 \theta(y) + \left(\frac{\partial X}{\partial y}\right)_0 y + \tilde{X}^*(\theta(y), 0, y) = 0 \qquad (118)$$

Consider now the change of variables $\mathrm{u} = x - \theta(y)$, $\mathrm{v} = \dot{x}$ in Eq. (115). Differentiation with respect to time t yields

$$\dot{\mathrm{u}} = \dot{x} - \dot{\theta}(y) = \dot{x} - \frac{\partial\theta}{\partial y}\dot{y} = \dot{x} - \frac{\partial\theta}{\partial y}f(\mathrm{u} + \theta(y), \mathrm{v}, y) := \dot{x} + \mathrm{U}(\mathrm{u}, \mathrm{v}, y) \qquad (119)$$

where we have used the second equation in Eq. (115) in the third equality, and

$$\dot{\mathrm{v}} = \ddot{x} = \left(\frac{\partial X}{\partial x}\right)_0 (x - \theta(y)) + \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \mathrm{v} + \left(\frac{\partial X}{\partial x}\right)_0 \theta(y) + \left(\frac{\partial X}{\partial y}\right)_0 y + \tilde{X}^*(\mathrm{u} + \theta(y), \mathrm{v}, y)$$

where we have used the first equation in Eq. (115). Noting Eq. (118), the right-hand side of the last equation can be simplified to

$$\begin{aligned}\dot{\mathrm{v}} &= \left(\frac{\partial X}{\partial x}\right)_0 \mathrm{u} + \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \mathrm{v} + \tilde{X}^*(\mathrm{u} + \theta(y), \mathrm{v}, y) - \tilde{X}^*(\theta(y), 0, y)\\ &= \left(\frac{\partial X}{\partial x}\right)_0 \mathrm{u} + \left(\frac{\partial X}{\partial \dot{x}}\right)_0 \mathrm{v} + \mathrm{V}(\mathrm{u}, \mathrm{v}, y)\end{aligned} \qquad (120)$$

where $\mathrm{V}(\mathrm{u}, \mathrm{v}, y) := \tilde{X}^*(\mathrm{u} + \theta(y), \mathrm{v}, y) - \tilde{X}^*(\theta(y), 0, y)$.

Using Eqs. (119) and (120), Eq. (115) can be rewritten in the new coordinates u, v, y, in first-order form as:

$$\frac{d}{dt}\begin{bmatrix}\mathrm{u}\\ \mathrm{v}\end{bmatrix} = \begin{bmatrix}0 & I_{n-m}\\ \left(\frac{\partial X}{\partial x}\right)_0 & \left(\frac{\partial X}{\partial \dot{x}}\right)_0\end{bmatrix}\begin{bmatrix}\mathrm{u}\\ \mathrm{v}\end{bmatrix} + \begin{bmatrix}\mathrm{U}(\mathrm{u},\mathrm{v},y)\\ \mathrm{V}(\mathrm{u},\mathrm{v},y)\end{bmatrix} := B\begin{bmatrix}\mathrm{u}\\ \mathrm{v}\end{bmatrix} + \begin{bmatrix}\mathrm{U}\\ \mathrm{V}\end{bmatrix}$$
$$\frac{dy}{dt} = f(\mathrm{u} + \theta(y), \mathrm{v}, y) := f^*(\mathrm{u}, \mathrm{v}, y) \qquad (121)$$

Note that matrix B is $(2n-2m)$ by $(2n-2m)$. In the new coordinates, we see that the equilibrium manifold (117) simplifies to the hyperplane

$$\{(\mathrm{u}, \mathrm{v}, y):\mathrm{u} = 0, \mathrm{v} = 0\}$$

More importantly, since $f(\bullet, 0, *) = 0$, we see that $f^*(0, 0, y) = 0$; also from Eq. (119), that $\mathrm{U}(0, 0, y) = 0$; and from the definition of V, that $\mathrm{V}(0, 0, y) = 0$. Thus, in system in Eq. (121), we have

$$\mathrm{U}(0, 0, y) = 0,\ \mathrm{V}(0, 0, y) = 0, f^*(0, 0, y) = 0$$

and according to the Lyapunov–Malkin theorem (see Appendix D), if all the eigenvalues of the $(2n-2m)$ by $(2n-2m)$ matrix B have negative real parts, then the solution $\mathrm{u} = 0$, $\mathrm{v} = 0$, $y = 0$ is stable with respect to all the variables $(\mathrm{u}, \mathrm{v}, y)$ and asymptotically stable with respect to the variables (u, v). Also, when a solution is close enough to (0, 0, 0) then $\lim_{t\to\infty}((\mathrm{u}(t), \mathrm{v}(t)) = (0, 0)$ and $\lim_{t\to\infty} y(t) = c$, where c is a constant.

We observe that the characteristic equation describing the eigenvalues of λ of the matrix B in Eq. (121) is

$$\Delta_3 = \det\left[\lambda^2 I_{n-m} - \lambda\left(\frac{\partial X}{\partial \dot{x}}\right)_0 - \left(\frac{\partial X}{\partial x}\right)_0\right] = 0 \qquad (122)$$

which we can compare with the expressions in Eqs. (113) and (114). Furthermore, if the $(2n-2m)$ roots of Eq. (113) (or Eq. (114), or Eq. (122)) have negative real parts then the matrix $\left(\frac{\partial X}{\partial x}\right)_0$ must be nonsingular, because its determinant is the product of the roots.

Noting that the eigenvalues of the linearized system about the equilibrium state are invariant with respect to changes of coordinates, we then have the following central result.

Main Result. Consider the n-DOF unconstrained mechanical system described in Eq. (1) in which the matrix $M > 0$ and the n-vector Q do not explicitly depend on time t. Let this system be subjected to the m independent constraints described by

$$\varphi(q) = 0,\ \varphi = [\varphi_1, \ldots, \varphi_h]^T,\ \psi(q, \dot{q}) = 0,\ \psi(q, 0) = 0,\ \psi = [\psi_1, \ldots, \psi_u]^T \tag{123}$$

with $\mathrm{Rank}(A) = \mathrm{Rank}((\partial\varphi/\partial q)^T, (\partial\psi/\partial\dot{q})^T)^T = m$.

The explicit equation that describes the motion of the constrained system is given in Eq. (5), and we shall denote it here, for brevity, as

$$M\ddot{q} = Q + Q^c := \mathrm{F}(q, \dot{q}),\quad \text{or } \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}\mathrm{F}(q, \dot{q}) \end{bmatrix},\ M(q) > 0 \tag{124}$$

Let $(q^*, 0)$ be an equilibrium state of the system, which may or may not be an isolated state.

Then, upon linearization of Eq. (124),

(1) the eigenvalues λ at the equilibrium state $(q^*, 0)$ are given by the characteristic equation, noting that $F(q^*, 0) = 0$, is

$$\det\left[\lambda^2 M - \lambda\frac{\partial F}{\partial\dot{q}} - \frac{\partial F}{\partial q}\right]\Bigg|_{(q^*,0)} = 0 \tag{125}$$

which has at least $2m$ roots that are zero. The roots of Eq. (125) are the same as the eigenvalues of the Jacobian matrix of the last column vector in Eq. (124) at the equilibrium state.

(2) If this characteristic equation has at least one root with a positive real part, then the equilibrium state $(q^*, 0)$ of Eq. (124) is conditionally unstable (and therefore also unstable) with respect to the invariant set $C = \{(q, \dot{q}){:}\varphi(q) = 0, (\partial\varphi/\partial q)\dot{q} = 0, \psi(q, \dot{q}) = 0\}$.

(3) If this characteristic equation has $2(n - m)$ roots with negative real parts, then the equilibrium state $(q^*, 0)$: (a) belongs to an u-dimensional manifold, and it is conditionally stable, though not asymptotically so, with respect to the set C. (b) Any motion on C sufficiently close to the equilibrium state tends to one of the states on the equilibrium manifold as $t \to \infty$.

This result applies to the velocity-dependent constraints ψ that are linear and/or nonlinear with respect to the velocities, $\dot{q}$, which may also include integrable ones. If $u = 0$ (the system is constrained only by the finite holonomic (positional) constraints: $\varphi(q) = 0$) and the condition under (3) above is satisfied, then the equilibrium is isolated and it is conditionally asymptotically stable.

Remark 2. When the characteristic equation (125) has no roots with positive real parts and the multiplicity of the zero root is greater than $2m$ and/or there are purely imaginary roots among the nonzero ones, the result is not applicable, and to examine the stability a nonlinear analysis must be carried out. In this case, it is probably simplest to apply the direct Lyapunov method, but, in general, there is no systematic procedure for determining the auxiliary function that this method requires, and its construction may be challenging.

Remark 3. Ref. [24] states a claim made by Aizerman and Gantmacher [26] that they have completely solved the problem of the stability of nonholonomic systems, which upon linearization have characteristic polynomials that have multiple zeros. In Ref. [26] (see, also [29]), the present authors, using Appel's equations, found a result in this regard, restricted to Pfaffian nonholonomic constraints. However, no detailed proof is provided in this reference, and to the best of the authors' knowledge, no complete proof appears to be currently available. As mentioned, our result is more general in scope than the claim in Ref. [26], and it is proved in detail. We shall employ this result to analyze the stability of dynamical systems in the examples that follow.

Example 8.

A. Consider the unconstrained system (11) described in Example 1, with the (simplified) holonomic constraint

$$\varphi_1(x_1, x_2, t) := x_1 - \frac{1}{2}x_2^2 = 0 \tag{126}$$

which is obtained from Eq. (12) by setting $g = 0$. The equations of motion in the coordinates (q_1, q_2) are given by Eq. (25) and we obtain

$$\begin{aligned} &\ddot{q}_1 = 0, && q_1(0) = 0,\ \dot{q}_1(0) = 0 \\ &\ddot{q}_2 = -\frac{1}{(1 + q_2^2)}(q_2\dot{q}_2^2 + k_2 q_2 + k_1\frac{1}{2}q_2^3), && q_2(0) = c_2,\quad \dot{q}_2(0) = c_4 \end{aligned} \tag{127}$$

The set of equilibrium states of this system is

$$\{(q_1, q_2, \dot{q}_1, \dot{q}_2){:}\dot{q}_1 = \dot{q}_2 = 0,\ q_1 = 0,\ q_2(k_2 + k_1 q_2^2/2) = 0\}$$

which contains the point $(0, q_2^{(1)} = 0, 0, 0)$. In addition, when $k_1k_2 < 0$, the equilibrium states include two points $(0, q_2^{(2,3)} = \pm\sqrt{-2k_2/k_1}, 0, 0)$, and when $k_1 = k_2 = 0$ the equilibrium states are continuously distributed along the q_2 axis.

Noting that an equilibrium state requires that we have $q_2(k_2 + k_1q_2^2/2) = 0$ the characteristic equation of the linearized system (127) about an equilibrium state is

$$\lambda^2[\lambda^2 + (k_2 + 3k_1q_2^2/2)/(1 + q_2^2)] = 0$$

and, according to the above stability Result, when $k_2 < 0$ the equilibrium state $(0, q_2^{(1)} = 0, 0, 0)$ is conditionally unstable, and when $k_1 < 0$ and $k_2 > 0$ the equilibrium states $(0, q_2^{(2,3)} = \pm\sqrt{-2k_2/k_1}, 0, 0)$ are conditionally unstable.

Although every solution of the differential equation system in Eq. (127) is unstable, it should be borne in mind that in the four-dimensional phase space the orbits of the constrained system lie on the invariant manifold $C := \{(q_1, q_2, \dot{q}_1, \dot{q}_2){:}q_1 = \dot{q}_1 = 0\}$, since $q_1(t) \equiv 0$. Then, it is obvious that the conditional stability (instability) of an equilibrium state of the system (127) is the same as the unconditional stability (instability) of the corresponding equilibrium state of the reduced (second) equation in Eq. (127), which allows us to advance in the study of the stability of this system. Indeed, the constrained system's Lagrangian is obtained by setting $g = 0$ in Eq. (31), which gives

$$L = \frac{1}{2}(1 + q_2^2)\dot{q}_2^2 - \frac{1}{2}(k_2q_2^2 + \frac{1}{4}k_1q_2^4) := T - V(q_2) \tag{128}$$

The first member on the right is the kinetic energy T, and the second member represents the potential energy, V.

According to the famous Lagrange–Dirichlet theorem, an equilibrium position of a conservative Lagrangian system is Lyapunovstable if the potential energy has a strict minimum at that position. What happens when this stability condition is not met is a classical, incompletely solved problem. However, for the system of one or two degrees-of-freedom with an analytic potential energy it is known that the condition of strict minimum is both sufficient and necessary for stability [37, 38]. In particular, for such systems, if an equilibrium position is nonisolated, it is unstable.

Since

$$\frac{\partial V}{\partial q_2} := V' = (k_2 + \frac{k_1}{2}q_2^2)q_2, \quad \text{and } \frac{\partial^2 V}{\partial q_2^2} := V'' = k_2 + \frac{3}{2}k_1q_2^2$$

for the reduced system, we conclude that:

(i) If $k_2 > 0 (k_2 < 0)$, the equilibrium state $q_2 = q_2^{(1)} = 0, \dot{q}_2 = 0$ is stable (unstable) because $V''(q_2^{(1)}) = k_2$.

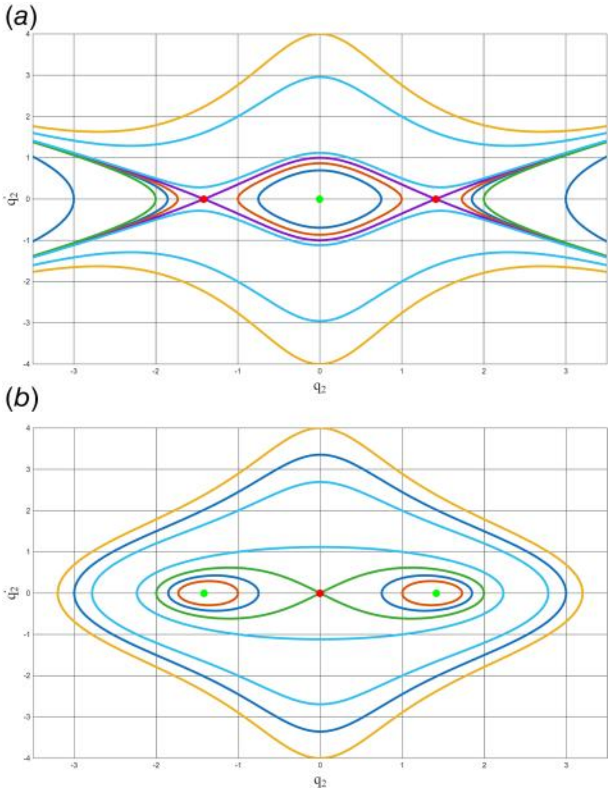

Fig. 4 (*a*) Phase portrait for $k_1 = -1, k_2 = 1$ and (*b*) phase portrait for $k_1 = 1, k_2 = -1$

(ii) If $k_2 = 0$, then $V = \frac{k_1}{8} q_2^4$, and consequently when $k_1 > 0 (k_1 < 0)$, the equilibrium state $q_2 = q_2^{(1)} = 0$, $\dot{q}_2 = 0$ is stable (unstable).

(iii) If $k_2 < 0$ and $k_1 > 0$ ($k_2 > 0$ and $k_1 < 0$) the equilibrium states $q_2 = q_2^{(2,3)}$, $\dot{q}_2 = 0$ are stable (unstable), since $V''(q_2^{(2,3)}) = -2k_2$.

(iv) When $k_1 = k_2 = 0$, then any equilibrium state $q_2 = a$, $\dot{q}_2 = 0$, where a is a (real) number, is unstable.

It should be noted that from the point of view of the equations in Eq. (127), for example, when $k_1, k_2 > 0$, the equilibrium position $q_1 = \dot{q}_1 = q_2 = \dot{q}_2 = 0$ is unstable. However, it is *conditionally stable* with respect to the constraint manifold C.

The unconstrained system in Eq. (11) has an equilibrium position at the origin ($x_1 = x_2 = 0$) that is unstable when k_1 and/or k_2 are negative. This equilibrium position continues to be an equilibrium position for the constrained system ($q_1 = q_2 = 0$). Yet the constrained system, as seen in (i) above, has a stable equilibrium point at the origin when $k_1 < 0$ and $k_2 > 0$, a nonintuitive result that shows that holonomic constraints can change the fundamental behavior of an unconstrained system. Moreover, the constrained system exhibits two more equilibrium positions, and for $k_1 > 0$ and $k_2 < 0$ these two positions are stable. Figure 4 shows the phase portrait of the reduced system in these cases. Stable equilibrium positions are shown by (green) dots, unstable ones by (red) dots.

The above (conditional) stability analysis is contingent upon the assumption that the system remains on the constraint manifold C. We do not numerically integrate the "shadow" equation, which is Lyapunov-unstable because of perturbations in q_1 and $\dot{q}_1$ that always arise during numerical simulations and simply assume that the system lies on C. However, as mentioned before, the 2-DOF constrained system described by the system of differential equations in Eq. (127) *is* always unstable because the first differential equation in it is unstable. Since an autonomous change of coordinates of a holonomic system does not change the character of the stability of the system, we can think of changing the coordinates back from $(q_1, q_2) \rightarrow (x_1, x_2)$, which indeed were the original coordinates. This leads us back to Eq. (18) (with $g = 0$), which must then also be an unstable system. We thus explain the difficulty we had earlier in obtaining the proper numerical response of the constrained holonomic systems (though somewhat intuitively, because the constraint there is nonautonomous) when computations are carried out in the (x_1, x_2) coordinate system, as we saw in Fig. 2. This difficulty in obtaining accurate computational results is, of course, a general characteristic of constrained holonomic systems, and it applies as well to n-DOF holonomically constrained systems.

In view of the above, it is generally beneficial, especially in numerical computations with larger systems that have more degrees-of-freedom, to cast the first-order form of the equations of motion carefully. The constraints can be used when expressing the velocities while eliminating the corresponding equations in the accelerations. For instance, in this example, the state vector can be chosen as $s = [x_1, x_2, \dot{x}_2]^T$ and the equations to be numerically integrated, cast as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2\dot{x}_2 \\ \dot{x}_2 \\ -(k_2x_2 + \dot{x}_2^3 + k_1x_1x_2)/(1 + x_2^2) \end{bmatrix} \tag{129}$$

where we observe that on the right-hand side, the first entry comes from the constraint (126), and on the left-hand side, $\dot{x}_1$ has been eliminated from the usual four-dimensional state vector. If needed, $\dot{x}_1$ can be numerically computed once $x_1(t)$ is computed. This idea can also be used to provide better accuracy in numerical simulations of dynamical systems that have holonomic and/or nonholonomic constraints.

B. Going from a 2-DOF system to a 3-DOF system—a modest increase in the number of degrees-of-freedom—with the unconstrained system (analogous to that in part A, above) described by

$$\ddot{x} := \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = -\begin{bmatrix} k_1x_1 \\ k_2x_2 \\ k_3x_3 \end{bmatrix} := Q,\ x(0) := \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} := x_0,$$

$$\dot{x}(0) := \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{bmatrix} = \begin{bmatrix} c_4 \\ c_5 \\ c_6 \end{bmatrix} := \dot{x}_0 \tag{130}$$

upon which we impose a holonomic constraint (analogous to that in part A, above) given by

$$\varphi_1(x_1, x_2, t) := x_1 - \frac{1}{2}(x_2^2 + x_3^2) = 0 \tag{131}$$

the complexity of the constrained system significantly increases, as we shall see.

The equation of motion of the constrained system, using Eq. (5), yields the system

$$\ddot{x} := \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = -\begin{bmatrix} k_1x_1 \\ k_2x_2 \\ k_3x_3 \end{bmatrix} + \frac{g(x)}{\Delta}\begin{bmatrix} 1 \\ -x_2 \\ -x_3 \end{bmatrix},$$

$$x(0) := \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} c'_1 \\ c_2 \\ c_3 \end{bmatrix} := x_0,\ \dot{x}(0) := \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{bmatrix} = \begin{bmatrix} c'_4 \\ c_5 \\ c_6 \end{bmatrix} := \dot{x}_0 \tag{132}$$

where $\Delta = 1 + x_2^2 + x_3^2$, $g(x) = \dot{x}_2^2 + \dot{x}_3^2 + k_1x_1 - k_2x_2^2 - k_3x_3^2$, $c'_1 = (c_2^2 + c_3^2)/2$, and $c'_4 = c_2c_4 + c_3c_6$. The latter two relations, which deal with the specification of the initial conditions, are necessitated by the imposition of the constraint.

Using the coordinate transformation $(x_1, x_2, x_3) \to (q_1, q_2, q_3)$, where $q_1 = \varphi_1(x_1, x_2, t) \equiv 0$, $q_2 = x_2$, and $q_3 = x_3$, the "reduced" system in first-order form is

$$\frac{d}{dt}\begin{bmatrix} q_2 \\ q_3 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ f_1/\Delta \\ f_2/\Delta \end{bmatrix} \tag{133}$$

where $f_1 = -[(k_1 q_2^3/2) + k_2 q_2 + (k_1/2 + k_2 - k_3) q_2 q_3^2 + q_2(\dot{q}_2^2 + \dot{q}_3^2)]$, $f_2 = -[(k_1 q_3^3)/2 + k_3 q_3 + (k_1/2 + k_3 - k_2) q_2^2 q_3 + q_3(\dot{q}_2^2 + \dot{q}_3^2)]$, and $\Delta = 1 + q_2^2 + q_3^2$. As shown in Appendix A, Eq. (133) is an unconstrained equation, i.e., its initial conditions can be arbitrarily prescribed; the constraints have already been incorporated when obtaining it, and, as we noted earlier, the conditional stability of the constrained system reduces to the stability of the "reduced" system.

By using Eq. (131) to eliminate $x_1(t)$ and $\dot{x}_1(t)$ from the Lagrangian of the unconstrained system (130), we obtain the Lagrangian of the reduced constrained system

$$L(q_2, q_3, \dot{q}_2, \dot{q}_3) = \frac{1}{2}[\dot{q}_2^2 + \dot{q}_3^2 + (q_2\dot{q}_2 + q_3\dot{q}_3)^2] - \frac{1}{2}[k_2 q_2^2 + k_3 q_3^2 + \frac{k_1}{4}(q_2^2 + q_3^2)^2] \tag{134}$$

The first member in the expression on the right-hand side is the kinetic energy, $T(q_2, q_3, \dot{q}_2, \dot{q}_3)$ of the constrained system and the second corresponds to its potential energy, $V(q_2, q_3)$. Note that the energy of the system, $H = T + V$, at the origin, i.e., at $q_i = \dot{q}_i = 0, \quad i = 2, 3$, is zero.

As seen from Eq. (133), all the equilibrium states of the constrained system require $\dot{q}_2 = \dot{q}_3 = 0$. In these relations that are applicable to all the equilibrium states, for different values of $k_1, k_2,$ and k_3, the equilibrium states are given by

(1) $q_2 = q_3 = 0,$,
(2) $q_2 = 0, q_3 = \pm\sqrt{-2k_3/k_1}, k_1 k_3 < 0,$
(3) $q_2 = \pm\sqrt{-2k_2/k_1}, q_3 = 0, k_1 k_2 < 0,$
(4) $q_3 = 0, k_1 = k_2 = 0,$
(5) $q_2 = 0, k_1 = k_3 = 0,$
(6) $k_1 = k_2 = k_3 = 0,$ and
(7) $q_2^2 + q_3^2 = -2k/k_1, k_2 = k_3 = k, kk_1 < 0.$

These equilibrium positions are the stationary (critical) points of the potential energy V, given by considering $dV = 0$. The first equilibrium state is isolated when either $k_2 k_3 \neq 0$ or $k_1 \neq 0$, and the equilibrium states (2) and (3) are isolated when $k_2 \neq k_3$; the remainder are nonisolated. In case (4), the equilibrium points are distributed along the q_2 axis; in case (5), along the q_3 axis; in case (6), over the (q_2, q_3) plane; and in case (7) along a circle of radius $\rho = \sqrt{-2k/k_1} > 0$ (with $kk_1 < 0$) in the (q_2, q_3) plane whose center is at the origin. Notice that the fixed points in (2) and (3) above lie on the circular ring in (7) when $k_2 = k_3 = k$; they lie along the q_3 and q_2 axes, respectively.

We next determine the stability of these equilibrium sets. As we know, an equilibrium state of this two degrees-of-freedom system can *only* be stable when it is isolated.

(1) The function V at (0, 0) has a strict minimum if and only if one of the following three conditions are satisfied: (a) $k_2 > 0, k_3 > 0$; (b) $k_2 = 0, k_3 \geq 0, k_1 > 0$; (c) $k_2 \geq 0, k_3 = 0, k_1 > 0$. In these cases, the isolated equilibrium state at the origin, namely, $(q_2, q_3, \dot{q}_2, \dot{q}_3) = (0, 0, 0, 0)$ is therefore stable. Otherwise, it is unstable; for example, if either $k_2 < 0$ or $k_3 < 0$.
(2) The Hessian matrix of $V(q_2, q_3)$ at $q_2 = 0, q_3 = \pm\sqrt{-2k_3/k_1}$, $k_1 k_3 < 0$ is $\mathrm{diag}(k_2 - k_3, -2k_3)$, and, noting that $k_2 \neq k_3$ for the isolated equilibrium position, it follows that a strict minimum of V is obtained only when $k_2 - k_3 > 0$ and $k_3 < 0$. Hence, the equilibrium point is Lyapunov-stable if and only if $k_2 - k_3 > 0$ and $k_3 < 0$.
(3) In a similar manner, at the equilibrium point $q_2 = \pm\sqrt{-2k_2/k_1}$, $q_3 = 0$, $k_1 k_2 < 0$, the system is stable if and only if $k_3 - k_2 > 0$ and $k_2 < 0$.

Although nonisolated equilibria of the system cannot be stable in the sense of Lyapunov, they may be stable in some other sense, which is treated below.

(4) From Eq. (134) the sum of T and V when $k_1 = k_2 = 0$ and can be written as

$$H(q_2, q_3, \dot{q}_2, \dot{q}_3) = \frac{1}{2}[\dot{q}_2^2 + \dot{q}_3^2 + (q_2\dot{q}_2 + q_3\dot{q}_3)^2] + \frac{1}{2}k_3 q_3^2$$

where H denotes the total energy of the system, which is conserved. Consider the equilibrium state $(\bar{q}_2, 0, 0, 0)$ where $\bar{q}_2$ is arbitrary, with $-\infty < \bar{q}_2 < \infty$. Let us denote $y := (q_3, \dot{q}_2, \dot{q}_3)$. Observe that: $H(\bar{q}_2, 0, 0, 0) = 0$, $\dot{H} \equiv 0$, and $H(q_2, q_3, \dot{q}_2, \dot{q}_3) \geq W(y) := \frac{1}{2}(k_3 q_3^2 + \dot{q}_2^2 + \dot{q}_3^2)$. Thus, when $k_3 > 0$, the function $H(q_2, q_3, \dot{q}_2, \dot{q}_3)$ is positive definite with respect to y, and H is said to be y-positive definite. Then the equilibrium state $(\bar{q}_2, 0, 0, 0)$ is y-stable, i.e., Lyapunov-stable with respect to the coordinate q_3, and the velocities $\dot{q}_2$ and $\dot{q}_3$ [39, 40].
(5) In the same manner as in (4) above, we conclude that when $k_2 > 0$, the equilibrium state $(0, \bar{q}_3, 0, 0)$ where $\bar{q}_3$ is arbitrary and $-\infty < \bar{q}_3 < \infty$ is Lyapunov-stable with respect to the coordinate q_2, and the velocities $\dot{q}_2$ and $\dot{q}_3$.
(6) It is easy to conclude that when $k_1 = k_2 = k_3 = 0$, any arbitrary equilibrium point $(\bar{q}_2, \bar{q}_3, 0, 0)$ with $-\infty < \bar{q}_2, \bar{q}_3 < \infty$ is unstable, though it is stable with respect to the velocities $\dot{q}_2$ and $\dot{q}_3$.
(7) When $k_2 = k_3 = k, f_1 = -q_2[k + k_1(q_2^2 + q_3^2)/2 + \dot{q}_3^2)]$ and $f_2 = -q_3[k + k_1(q_2^2 + q_3^2)/2 + (\dot{q}_2^2 + \dot{q}_3^2)]$ in Eq. (133). The set of equilibrium points form a circle in the (q_2, q_3)-plane that is centered at the origin and has a radius $\rho = \sqrt{-2k/k_1} > 0$; recall, $kk_1 < 0$.

The presence of the circular ring motivates a transformation to polar coordinates, and we use the variables x and $0 \leq \theta < 2\pi$ by setting

$$q_2 = (\rho + x)\cos(\theta), \; q_3 = (\rho + x)\sin(\theta), \; x \neq -\rho \tag{135}$$

to study the stability of this ring of nonisolated equilibrium points.

In the polar coordinates used, the circle of nonisolated equilibrium points is described by $x = 0, \dot{x} = 0, \theta$ arbitrary, and $\dot{\theta} = 0$. Since $\dfrac{\partial(q_2, q_3)}{\partial(x, \theta)} \neq 0, x \neq -\rho$, there is a one-to-one mapping $(q_2, q_3, \dot{q}_2, \dot{q}_3) \to (x, \dot{x}, \theta, \dot{\theta})$. Given the values of q_2 and q_3, Eq. (135) can be used to obtain the corresponding values of x and θ; given values of $\dot{q}_2$ and $\dot{q}_3$, the values of $\dot{x}$ and $\dot{\theta}$ can be obtained from

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\dfrac{\sin(\theta)}{\rho + x} & \dfrac{\cos(\theta)}{\rho + x} \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{136}$$

Using this coordinate transformation, the Lagrangian in Eq. (134), with $k_2 = k_3 = k$, becomes

$$L(x, \theta, \dot{x}, \dot{\theta}) = \frac{1}{2}[\dot{x}^2 + (\rho + x)^2\dot{x}^2 + (\rho + x)^2\dot{\theta}^2] - \frac{(\rho + x)^2}{2}[k + \frac{k_1}{4}(\rho + x)^2] \tag{137}$$

The first member on the right-hand side is the kinetic energy, T while the second member corresponds to the potential energy, V. Observe the absence of θ from the Lagrangian, showing that it is rotationally invariant. Since $\frac{\partial L}{\partial \theta} = 0$, the angular momentum of the constrained system is conserved along any orbit, and we have

$$\frac{\partial L}{\partial \dot{\theta}} = (\rho + x)^2 \dot{\theta} = q_2 \dot{q}_3 - q_3 \dot{q}_2 = c \tag{138}$$

where c is a constant.

Using the Lagrangian in Eq. (137) and noting that $\rho^2 k_1 = -2k$, $kk_1 < 0$, we get the following second-order equations of motion

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = - \begin{bmatrix} \dfrac{(\rho + x)}{1 + (\rho + x)^2} (k_1 \rho x + \dfrac{k_1 x^2}{2} + \dot{x}^2 - \dot{\theta}^2) \\ \dfrac{2\dot{x}\dot{\theta}}{(\rho + x)} \end{bmatrix} \tag{139}$$

The set of equilibrium points lies on the manifold

$$M = \{(x, \theta, \dot{x}, \dot{\theta}\} : x = 0, \dot{x} = 0, \dot{\theta} = 0\} \tag{140}$$

which is an invariant set. We note that the invariant set M is stable if for all $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that when $\|\theta(0)\|$ is arbitrary and $\|x(0), \dot{x}(0), \dot{\theta}(0)\| < \delta$, then $\|x(t), x(t), \dot{\theta}(t)\| < \delta$ for $t \geq 0$.

Since the variable θ does not appear in Eq. (139), the stability of the set M is the same as the stability of the trivial solution of the dynamical system

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\dfrac{(\rho + x)}{1 + (\rho + x)^2} \left(k_1 \rho x + \dfrac{k_1 x^2}{2} + \dot{x}^2 - \dot{\theta}^2 \right) \\ -\dfrac{2\dot{x}\dot{\theta}}{(\rho + x)} \end{bmatrix} \tag{141}$$

We consider now the function

$$\begin{aligned} H(x, \dot{x}, \dot{\theta}) = {} & \frac{1}{2} [(1 + (\rho + x)^2) \dot{x}^2 + (\rho + x)^2 \dot{\theta}^2] \\ & + \frac{1}{2} \left[k(\rho + x)^2 + \frac{k_1}{4} (\rho + x)^4 - k\rho^2 - \frac{k_1}{4} \rho^4 \right] \end{aligned} \tag{142}$$

which has the following two properties:

$$H(0, 0, 0) = 0$$

$$\left. \frac{dH}{dt} \right|_{(141)} = \dot{x}(\rho + x)\left(\frac{k_1 \rho^2}{2} + k\right) = 0$$

In the second property, the last equality follows because $\rho = \sqrt{-2k/k_1}$, with $kk_1 < 0$. Moreover, using this value of ρ, H in Eq. (142) can be rewritten as follows

$$\begin{aligned} H(x, \dot{x}, \dot{\theta}) = {} & \frac{1}{2} [-2kx^2 + (1 + (\rho + x)^2) \dot{x}^2 + (\rho + x)^2 \dot{\theta}^2] \\ & + \left[\frac{k_1 \rho x^3}{2} + \frac{k_1 x^4}{8} \right] \end{aligned} \tag{143}$$

We see from the first member on the right-hand side that when $k < 0$, then $H(x, \dot{x}, \dot{\theta})$ is positive definite in the neighborhood of the point (0, 0, 0), making it a suitable Lyapunov function. Being positive definite, we have proved that the trivial solution is then Lyapunov-stable.

Furthermore, the Jacobian of the right-hand side of Eq. (141) with respect to the column vector $[x, \dot{x}, \dot{\theta}]^T$ evaluated at (0,0,0) is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha = -\frac{k_1 \rho^2}{(1 + \rho^2)}$$

and its eigenvalues are $\{0, \sqrt{-k_1} \frac{\rho}{\sqrt{1+\rho^2}}, -\sqrt{-k_1} \frac{\rho}{\sqrt{1+\rho^2}}\}$. Hence, the point (0,0,0) is Lyapunov-unstable when $k_1 < 0$.

In summary, when $k_2 = k_3 = k < 0$ (and $k_1 > 0$, since $kk_1 < 0$), the set of all the nonisolated equilibrium points, which lie on the circle of radius $\rho = \sqrt{-2k/k_1}$, with center at the origin of the (q_2, q_3)-plane, is stable; when $k_1 < 0$ (and therefore, $k > 0$), this set is unstable.

Using (1) and (7) above, we conclude the following when $k_2 = k_3 = k$ and $kk_1 < 0$:

(i) When $k < 0$ (and $k_1 > 0$) then the origin is unstable, while the circle of nonisolated equilibrium points with center at the origin and radius $\rho = \sqrt{-2k/k_1}$ is stable.
(ii) When $k > 0$ (and $k_1 < 0$) then the origin is stable, while the nonisolated equilibrium points that lie on a circle with center at the origin and radius ρ are all unstable.

Before we exhibit the dynamics of this constrained system we analytically consider orbits of the system that asymptotically tend to unstable equilibrium states, for example, at the origin $q_2 = q_3 = \dot{q}_2 = \dot{q}_3 = 0$, when $k < 0$. At an equilibrium state, we require $\dot{q}_2 = \dot{q}_3 = 0$, so that $c = 0$ in Eq. (138), and $\theta =$ a constant. Since Eq. (138) is an integral of the motion, all the orbits that tend to (or away from) an equilibrium state in the phase space $\{q_2, q_3, \dot{q}_2, \dot{q}_3\}$ therefore lie in two-dimensional planes described by

$$q_2 = (\rho + x)\cos(\theta),\ q_3 = (\rho + x)\sin(\theta),\ \dot{q}_2 = \dot{x}\cos(\theta),\ \dot{q}_3 = \dot{x}\sin(\theta) \tag{144}$$

where $0 \leq \theta < 2\pi$ is an arbitrary, but fixed, parameter.

From Eq. (137), we see that the energy integral, $H = T + V$, of the constrained system along orbits that lie in the plane (144), becomes ($\dot{\theta} = 0$)

$$H^*(x, \dot{x}) = \underbrace{\frac{1}{2}[1 + (\rho + x)^2]\dot{x}^2}_{T^*} + \underbrace{\frac{(\rho + x)^2}{2}[k + \frac{k_1}{4}(\rho + x)^2]}_{V^*} = h \tag{145}$$

The asymptotic orbits lie on the critical levels of the function H^*, which are $h = h^* = 0$ when $k < 0$, and $h^* = -k^2/(2k_1)$ when $k > 0$. The first value of h^* relates to orbits that leave the unstable equilibrium position at the origin of the four-dimensional phase space $\{q_2, q_3, \dot{q}_2, \dot{q}_3\}$ when $k < 0$ (with $k_1 > 0$), and it is obtained by setting $x = -\rho$ and $\dot{x} = 0$ in Eq. (145). The second value of h^* relates to each of the nonisolated unstable equilibrium positions on the ring of radius ρ around the origin when $k > 0$ (with $k_1 < 0$), and it is obtained by setting $x = 0$, and $\dot{x} = 0$, $\rho = \sqrt{-2k/k_1}$, in Eq. (145). Using Eq. (145), we next sketch the phase portraits in the $(x, \dot{x})$-plane for these two cases.

A. When $k < 0$, $k_1 > 0$: From Eq. (145), taking into account that $h = h^* = 0$, and denoting $\Delta := [1 + (\rho + x)^2]$, we obtain

$$\dot{x} = \pm(\rho + x)\sqrt{-[k + \frac{k_1}{4}(\rho + x)^2]/\Delta} \tag{146}$$

In Fig. 5, we sketch the potential energy V^* shown in (145) and the orbits of the dynamical system using $k = -4$, $k_1 = 2$, so that $\rho = 2$. In the $(x, \dot{x})$-plane, the unstable equilibrium point is located at $(-2, 0)$ (or $q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0$, in phase space); it is shown by the red dot. For any fixed value of θ, the ring of nonisolated equilibrium states, all of which are stable, intersects this plane at the points $(-4, 0)$ and $(0, 0)$, shown by the green dots.

The homoclinic orbit(loop) to the right of the unstable equilibrium point (red dot) extends over $-\rho < x \leq (\sqrt{2} - 1)\rho$ and the one to the left of the red dot extends over $-(\sqrt{2} + 1)\rho < x \leq -\rho$. As stated before, the value $h = h^* = 0$ corresponds to the unstable equilibrium point (red dot) and to the two homoclinic orbits that leave it and asymptotically tend to the red dot as $t \to \infty$. The value of h (in (145)) on each of the (colored) periodic orbits that lie within the homoclinic orbits(loops) decreases as we approach the two green dots (the ring) where it becomes $h = h^* = -k^2/(2k_1)$. The value of h is positive for orbits (black) outside of the homoclinic orbit. In the phase space $\{q_2, q_3, \dot{q}_2, \dot{q}_3\}$ for any fixed value of the parameter θ, $0 \le \theta < \pi$, the orbit is determined by using (144) in which $\dot{x}$ is given by (146).

Thus, somewhat amazingly, there exists for each value of $0 \le \theta < \pi$ a homoclinic orbit (this refers to both loops in the $(x, \dot{x})$-plane). Each such homoclinic orbit loops around the stable ring of nonisolated equilibrium points. We have a continuous distribution (for each continuously varying value of θ) of these homoclinic loops emanating from the origin. One gets a kind of "surface" created by the continuous distribution of homoclinic orbits around the origin, which encloses the circle of nonisolated equilibrium positions.

A naïve approach through linearization of the equations of motion in (133) about the origin of the phase space $\{q_2, q_3, \dot{q}_2, \dot{q}_3\}$ would reveal that when $-a^2 := k < 0$, we get two uncoupled equations $\ddot{q}_2 - a^2 q_2 = 0$ and $\ddot{q}_3 - a^2 q_3 = 0$, showing that the origin is a saddle point, and that in the vicinity of the origin these saddles lie in the $(q_2, \dot{q}_2)$-plane and the $(q_3, \dot{q}_3)$-plane. Considered as a dynamical system in the four-dimensional q-space its eigenvalues are $\{a, a, -a, -a\}$ and the corresponding eigenvectors are the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ a & 0 & -a & 0 \\ 0 & a & 0 & -a \end{bmatrix}$$

Such an approach, while suggestive of complex behavior, would likely obscure the underlying simplicity and elegance of the dynamics revealed by the preceding analysis.

To illustrate the dynamics consider a system (133) again taking $k_2 = k_3 = k = -4$, and $k_1 = 2$, so that $\rho = 2$. We expect to see an unstable origin surrounded in the (q_1, q_2)-plane by a circle of radius ρ, showing a kind of geometrical symmetry. We numerically compute an orbit that starts in the neighborhood of the stable circular ring of nonisolated equilibrium points. To bring out the point of showing this, the computation is carried out for only 16 secs. The initial conditions at $t = 0$ for the orbit near the circular ring are

$$q_2 = (\rho + \varepsilon_1)\cos\theta_0,\ q_3 = (\rho + \varepsilon_1)\sin\theta_0,\ \dot{q}_2 = \varepsilon_2,\ \dot{q}_3 = \varepsilon_3$$

and those for the orbit near the origin are

$$q_2 = \varepsilon_4,\ q_3 = \varepsilon_5,\ \dot{q}_2 = \varepsilon_6,\ \dot{q}_3 = \varepsilon_7$$

Figure 6 shows the orbits for the values $\theta_0 = \pi/6$, $\varepsilon_1 = 0.2$, the value of all the remaining ε_i's being equal to zero. The circular ring of equilibrium points is shown in black. The start of each orbit is denoted by a small solid sphere. The stable periodic orbit that starts in the vicinity of the circular ring is shown in green. We see that the orbits (magenta, red, and blue) that leave the vicinity of the unstable origin are homoclinic. These orbits appear to be the four-dimensional counterparts of the homoclinic orbit in Fig. 5. We show just a few of them. The points on the homoclinic orbits satisfy the energy equation

$$\frac{1}{2}[(1 + q_2^2)\dot{q}_2^2 + (1 + q_3^2)\dot{q}_3^2 + 2q_2q_3\dot{q}_2\dot{q}_3] + \frac{1}{8}(q_2^2 + q_3^2)[k_1(q_2^2 + q_3^2) + 4k] = 0$$

For convenience, homoclinic orbits in a plane are shown by dashed and solid lines. The orbit with the starting point (green sphere), which is in the neighborhood of the ring, shows stable periodic behavior. Since the system has a four-dimensional phase space, for the sake of visualization, two 3-dimensional views are shown—in one, the vertical axis is $\dot{q}_2$, and in the other, $\dot{q}_3$.

Note that the blue straight line in Fig. 6(*a*) is a homoclinic orbit as seen in Fig. 6(*b*). These plots conform to our images of stable and unstable equilibrium points. Often, one has the intuitive notion that stability of an equilibrium point means "that the orbit remains close to the point it starts from, when starting "near" the equilibrium point."

Keeping all the constants the same as in Fig. 6, except for $\varepsilon_2 = 0.02$, we obtain the following phase portrait (see Fig. 7).

Observe that the orbit in green that starts very nearly from the same point as in Fig. 6 moves perceptibly, despite the short time over which the integration is done, spiraling along the (black) circular ring of equilibrium positions, which were shown to be stable for $k < 0$, from (143). At first sight, this does not comport with our image of a stable orbit. Over time, the (green) spiraling orbit keeps going around the circular ring, getting farther away from its starting point. This paradox gets resolved when we recall that we only proved that the fixed (equilibrium) points on the circle were stable with respect to the coordinates x, $\dot{x}$, and $\dot{\theta}$, which in fact they are, as confirmed by integrating (141). Noting from (136) that $\dot{q}_2(t)$ and $\dot{q}_3(t)$ affects both $\dot{x}(t)$ and $\dot{\theta}(t)$ one can visualize the spiraling orbit as forming a kind of (spiral) tube that gradually envelops the black circle of nonisolated equilibrium points, with the radius of the tube (x) and its rate of change $(\dot{x})$ both changing, as well as the rate of motion $(\dot{\theta})$ of the trajectory along the circle. It is these variables that are stable about the point $x = \dot{x} = \dot{\theta} = 0$, which was proved. This example points out the importance of being aware of the coordinates with respect to which a dynamical system is stable,

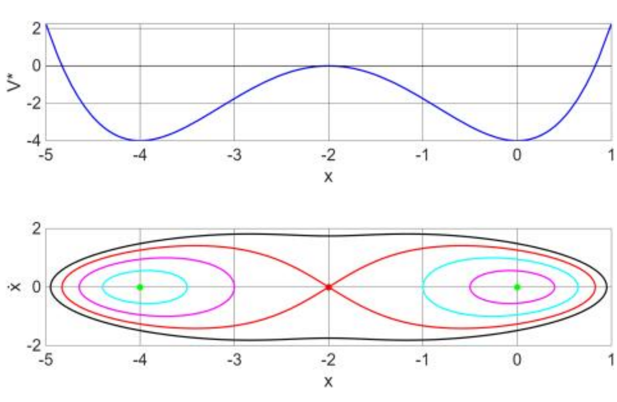

Fig. 5 Phase portrait showing the unstable equilibrium position (red dot) in the $(x, \dot{x})$-plane and two homoclinic orbits (in red) on either side of it

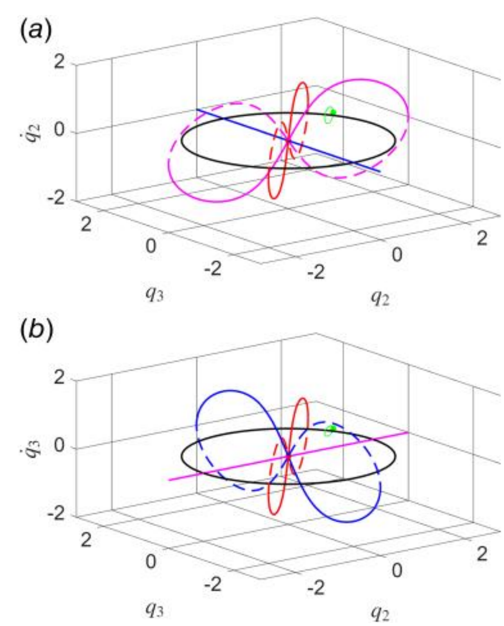

Fig. 6 Phase plots of the dynamical system: $k_1 = 2$, $k_2 = k_3 = k = -4$, $\rho = 2\theta_0 = \pi/3$, $\varepsilon_1 = 0.2$

something often forgotten when observing the motion of an orbit.

B. When $k > 0$ and $k_1 < 0$: From Eq. (145), taking into account that $h = h^* = -2k^2/(2k)$, and denoting $\Delta := [1 + (\rho + x)^2]$, we obtain

$$\dot{x} = \pm\sqrt{-\frac{k^2/k_1 + (\rho + x)^2[k + k_1(\rho + x)^2/4]}{\Delta}}, \; -2\rho < x < 0 \quad (147)$$

In Fig. 8 we sketch the potential energy V^* shown in Eq. (145) and the orbits of the dynamical system using $k = 4$, $k_1 = -2$, so that $\rho = 2$. In the $(x, \dot{x})$-plane, the stable equilibrium position is located at $(-2, 0)$ (or $q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0$, in the q-phase space); it is shown by the green dot. For any fixed value of θ, $0 \le \theta < \pi$, the ring of nonisolated equilibrium states, all of which are unstable, intersects this plane at the points $(-4, 0)$ and $(x = 0, \dot{x} = 0)$, shown by the red dots. We see that two heteroclinic orbits (shown in red) connect the unstable equilibrium points when $h^* = -2k/k_1 = 4$. They are separatrices, and stable periodic orbits reside inside these heteroclinic "loops"; we show one of them (cyan).

Orbits entering the unstable equilibrium positions are shown in green, while those leaving are shown in black. Every two equilibrium positions on the ring (of nonisolated equilibrium positions) that are diametrically opposite each other have two heteroclinic orbits connecting them. In the phase space $\{q_2, q_3, \dot{q}_2, \dot{q}_3\}$ for any fixed value of the parameter θ, $0 \le \theta < \pi$, the orbit is determined by using Eq. (144) in which $\dot{x}$ is given by Eq. (147).

It is worth comparing Figs. 4(*b*) and 5, and Figs. 4(*a*) and 8. The structure of the phase plots appears similar, yet there is a substantial difference between the dynamical behavior of the 2-DOF system and the 3-DOF system because in Figs. 5 and 8 this structure occurs at every value of $0 \le \theta < \pi$. More importantly, in Fig. 4 we have three isolated fixed points, while in Figs. 5 and 8 we have ring of nonisolated fixed points and an isolated fixed point.

For the parameters $k = 4$ and $k_1 = -2$, the structure of the phase portrait in the q-coordinates is shown in Fig. 9. Two pairs of points (red dots), each pair diametrically opposite the other, are shown on the ring (black) of unstable nonisolated equilibrium points. The two heteroclinic orbits between each pair are shown in red. A periodic orbit within each pair of these heteroclinic "loops" is also shown (green and cyan). *Every* such pair of points that are diametrically opposite each other on the ring has this orbit structure. The open orbit (blue) lies outside the heteroclinic loop that is in the plane of the green periodic orbit.

Example 9.

A. Consider an unconstrained system described by the equation

$$\ddot{x} = 0, \; \ddot{y} = 1, \; \ddot{z} = 0 \quad (148)$$

with the initial conditions $x(0) = a$, $y(0) = b$, $z(0) = c$, $\dot{x}(0) = p$, $\dot{y}(0) = q$, $\dot{z}(0) = r$, that is subjected to the nonholonomic constraint

$$\psi := \dot{y} - (x^2 + y^2 + z^2)\dot{x} = 0 \quad (149)$$

Using Eq. (5), the equation of motion of the constrained system is

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} (x^2 + y^2 + z^2)[1 - 2\dot{x}(x\dot{x} + y\dot{y} + z\dot{z})] \\ (x^2 + y^2 + z^2)^2 + 2\dot{x}(x\dot{x} + y\dot{y} + z\dot{z}) \\ 0 \end{bmatrix} \quad (150)$$

where $\Delta = 1 + (x^2 + y^2 + z^2)^2$.

The unconstrained system has no equilibrium position. For the constrained system, at the equilibrium position, we require that $\dot{x} = \dot{y} = \dot{z} = 0$. Using Eq. (150), we then find that the system has one unique, isolated equilibrium position at $x = y = z = \dot{x} = \dot{y} = \dot{z} = 0$. The equilibrium position is clearly unstable, since the third relation in Eq. (150) has the unbounded solution $z(t) = c + rt$, where c and r can be arbitrarily small numbers.

This contradicts the prevalent belief that a nonholonomically constrained system has only nonisolated equilibrium states (see, for instance, Ref. [24], Ch. 5, Sec. 2, p. 263]).

Furthermore, linearization of the right-hand side of Eq. (150) gives the characteristic equation $\lambda^6 = 0$ at this isolated equilibrium position since the right-hand side of Eq. (150) has no linear terms. This contradicts the conventional notion that it is the nonisolated fixed points in nonholonomic systems that are responsible, upon linearization, for the existence of zeros in their characteristic equations, as, for example, also stated in the above reference.

B. Consider first the unconstrained, linear, damped system in Example 5 that is given in Eq. (84) in which

$$V_x = a_x x, \; V_y = a_y y, \quad \text{and } V_z = a_z z \quad (151)$$

where $a_x, a_z > 0$, and $\delta_x, \delta_z > 0$. When $a_y \neq 0$, the dynamical system has a unique equilibrium state $x = y = z = \dot{x} = \dot{y} = \dot{z} = 0$, which is stable (asymptotically stable) if and only if $a_y > 0$, $\delta_y \ge 0$ $(a_y, \delta_y > 0)$.

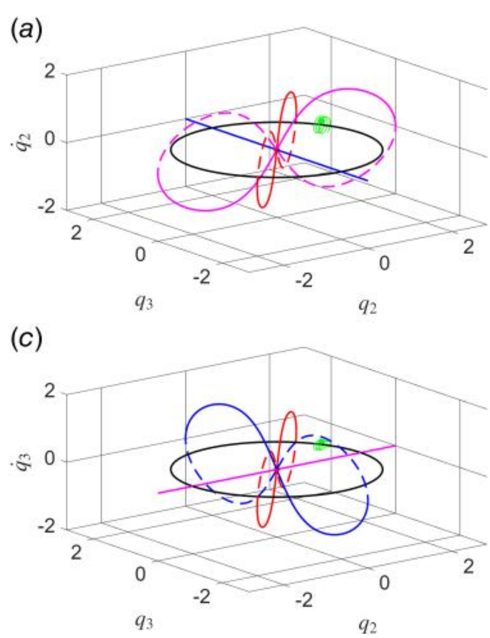

Fig. 7 A small portion of the spiral orbit of the dynamical system is shown in green. The orbit, in time, goes around the circular ring in a spiral motion, continuously traversing the entire perimeter of the circle of non-isolated equilibrium positions.

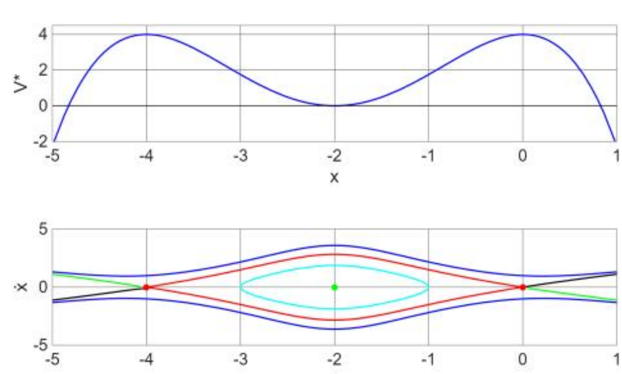

Fig. 8 Phase portrait showing the heteroclinic orbits between any two unstable nonisolated equilibrium positions that are diametrically opposite one another on the ring

We next subject this unconstrained linear system to the nonholonomic constraint (85) with $f(z) = z$ and $\alpha = 1$, which is

$$\psi := \dot{y} - z\dot{x} = 0 \tag{152}$$

The equation of motion (86) of the constrained system, then becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -(a_x x + a_y yz + z\dot{x}\dot{z} + \delta_x \dot{x} + \delta_y z\dot{y})/(1+z^2) \\ (\dot{x}\dot{z} - a_x xz - a_y yz^2 - \delta_x z\dot{x} - \delta_y z^2\dot{y})/(1+z^2) \\ -a_z z - \delta_z \dot{z} \end{bmatrix} \tag{153}$$

The set of equilibrium states of this system in $\mathbb{R}^3\{x, y, z\} \times \mathbb{R}^3\{\dot{x}, \dot{y}, \dot{z}\}$ lie on the line L described by the set

$$\{(x, y, z, \dot{x}, \dot{y}, \dot{z}) : x = z = 0,\ \dot{x} = \dot{y} = \dot{z} = 0,\ -\infty < y < \infty\}$$

Upon linearization of Eq. (153), we find that the characteristic equation at any point on L is

$$\lambda^2 p(\lambda) = 0,\ p(\lambda) = (\lambda^2 + \delta_x \lambda + a_x)(\lambda^2 + \delta_z \lambda + a_z)$$

Observe that two of the roots are zero, while the remainder of them have negative real parts since $a_x, a_z, \delta_x, \delta_z$ are each greater than zero. Thus, according to the stability Result proved at the beginning of this section, every point on the equilibrium line L is conditionally stable, and any motion of the constrained system with initial conditions sufficiently close to the equilibrium line tends to one state on this line when $t \to \infty$.

Observe that the constants a_y and δ_y do not appear in the characteristic equation, and the stability of the constrained system continues to hold even when one or both of these constants are negative. In that case, while the unconstrained system becomes unstable, the constrained system remains stable.

C. We next consider the nonlinear, damped unconstrained system described by Eq. (84) in Example 5 with

$$V_x = a_x x + b_x x^3,\ V_y = a_y y, \quad \text{and } V_z = a_z z + b_z z^3 \tag{154}$$

where we assume that the known constants in the a's, b's, and δ's are each arbitrary and *not* equal to zero, as also α. V_x is the partial derivative of V with respect to x, etc. Recall, the constraint is $\psi := \dot{y} - \alpha f(z)\dot{x} = 0,\ \alpha \neq 0$.

The equations of motion of the constrained nonholonomic system, using Eq. (85), then become

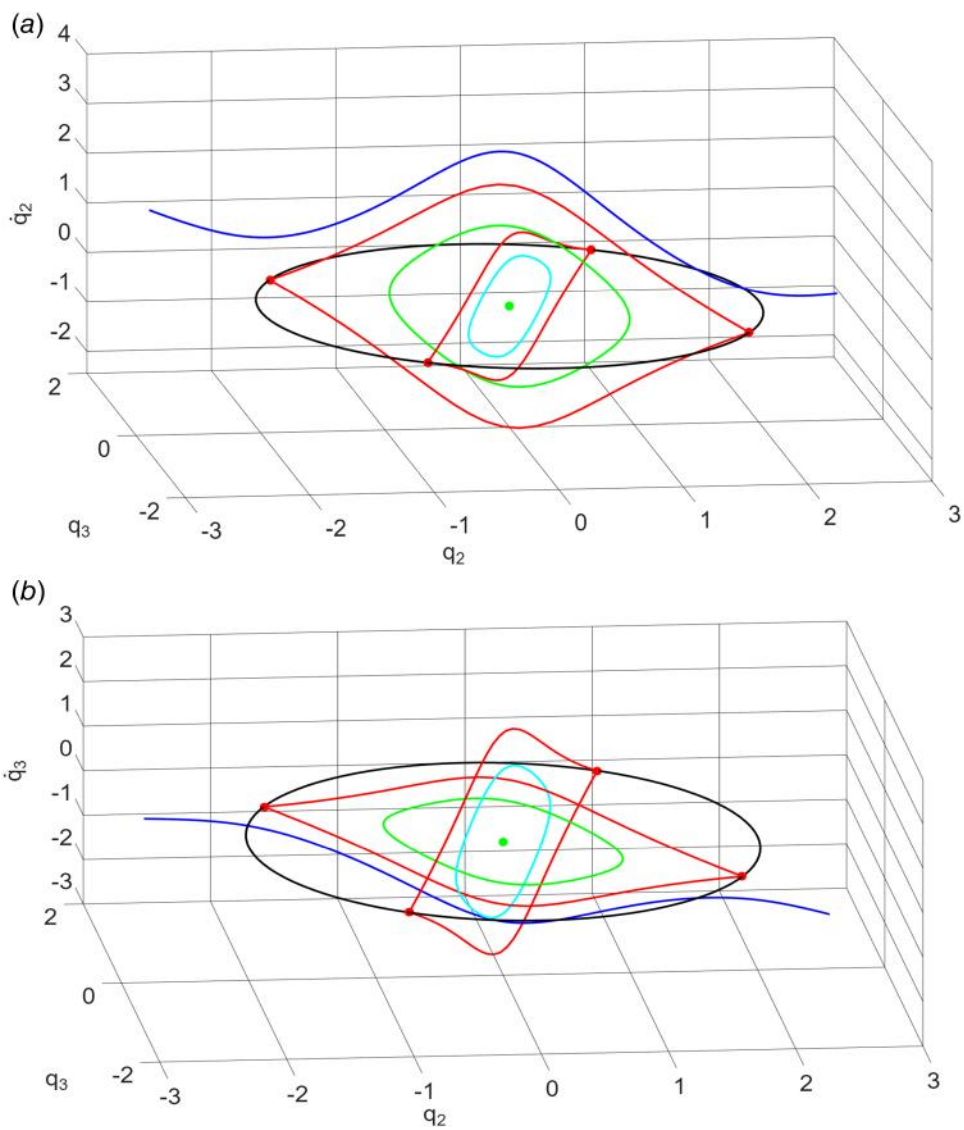

Fig. 9 Structure of the phase portrait in *q*-space for $k = 4, k_1 = -2$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -\begin{bmatrix} [a_x x + b_x x^3 + \delta_x \dot{x} + \alpha a_y f(z) y + \alpha \delta_y f(z) \dot{y} + \alpha^2 f(z) f'(z) \dot{x} \dot{z}]/\Delta \\ [\alpha f(z)(a_x x + b_x x^3 + \delta_x \dot{x} + \alpha a_y f(z) y + \alpha \delta_y f(z) \dot{y}) - \alpha f'(z) \dot{x} \dot{z}]/\Delta \\ a_z z + b_z z^3 + \delta_z \dot{z} \end{bmatrix} \tag{155}$$

with $x(0) = a$, $y(0) = b$, $z(0) = c$, $\dot{x}(0) = p$, $\dot{y}(0) = \alpha p f(c)$, $\dot{z}(0) = r$. Here, $\Delta = 1 + \alpha^2 f(z)^2 > 0$.

The set of equilibrium states of this system is

$$\{(x, y, z, \dot{x}, \dot{y}, \dot{z}) : \dot{x} = 0,\ \dot{y} = 0,\ \dot{z} = 0,\ z(a_z + b_z z^2) = 0, \\ x(a_x + b_x x^2) + \alpha a_y f(z) y = 0\} \tag{156}$$

After linearization, the characteristic equation of Eq. (155) at a point of this set is found to be

$$\lambda^2[\lambda^2 + \delta_z \lambda + g_1(z)][\lambda^2 + (g_2(z)/\Delta(z))\lambda + g_3(x, z)/\Delta(z)] = 0 \tag{157}$$

where

$$g_1 = a_z + 3b_z z^2, \\ g_2 = (\delta_x + \alpha^2 \delta_y f(z)^2), \quad \text{and} \\ g_3 = (a_x + 3b_x x^2 + \alpha^2 a_y f(z)^2) \tag{158}$$

Thus, according to the stability result, if all δ_z, g_1, g_2, and g_3 are positive, the corresponding equilibrium state is conditionally stable, and if at least one of δ_z, g_1, g_2, and g_3 is negative, then the state is conditionally unstable.

A point of the equilibrium set requires that $z(a_z + b_z z^2) = 0$, and we have the following two cases: (1) when $z = 0$, and (2) when $a_z b_z < 0$ so that $(a_z + b_z z^2) = 0$ for $z = z^{*(1)} := \sqrt{-a_z/b_z}$ and $z = z^{*(2)} := -\sqrt{-a_z/b_z}$.

(1) When $z = 0$: From Eq. (156), we require $yf(0) = -\dfrac{x(a_x + b_x x^2)}{\alpha a_y}$ to obtain the equilibrium states. Thus,

(a) if $f(0) = 0$, then the points along the line

$$\dot{x} = \dot{y} = \dot{z} = 0,\ x = 0,\ z = 0,\ -\infty < y < +\infty \tag{159}$$

are nonisolated equilibrium states. From (158), at each of these points $g_1 = a_z$, $g_2 = \delta_x$, $g_3 = a_x$, and it follows, using (157), that if all the constants δ_z, a_z, δ_x, a_x are positive (at least one of these constants is negative), then every point on the equilibrium line (159) is conditionally stable (unstable), irrespective of the values of a_y, δ_y, b_x, b_z and α.

(b) if $f(0) = 0$ and $a_x b_x < 0$, then the points

$$\dot{x} = \dot{y} = \dot{z} = 0, x = x^{*(1)} = \sqrt{-a_x/b_x}, z = 0, -\infty < y < +\infty \tag{160}$$

$$\dot{x} = \dot{y} = \dot{z} = 0, x = x^{*(2)} = -\sqrt{-a_x/b_x}, z = 0, -\infty < y < +\infty \tag{161}$$

are also (additional) nonisolated equilibrium states. For these equilibrium lines, we find from Eq. (158) that $g_1 = a_z$, $g_2 = \delta_x$ and $g_3 = -2a_x$. Using the characteristic equation (157), we then see that when $\delta_z > 0, a_z > 0$, $\delta_x > 0$, and $a_x < 0$ (at least one of δ_z, a_z, δ_x is negative and/or $a_x > 0$) then every point on the equilibrium lines (160) and (161) is conditionally stable (unstable), irrespective of the values of a_y, δ_y, b_z and α.

On the other hand,

(c) if $f(0) \neq 0$, then from

$$\dot{x} = \dot{y} = \dot{z} = 0,\ z = 0,\ y = -\frac{x(a_x + b_x x^2)}{\alpha a_y f(0)},\ -\infty < x < +\infty \tag{162}$$

form a set of nonisolated equilibrium states; at the point $y = 0$, the equilibrium position is the origin, and, if $a_x b_x < 0$, then we also have equilibrium positions at $x = x^*(i)$, $y = 0$, $z = 0$, $i = 1, 2$. In this case we find from Eq. (158) that $g_1 = a_z$, $g_2(0) = \delta_x + \alpha^2 \delta_y f(0)^2$, and $g_3(x, 0) = a_x + 3b_x x^2 + \alpha^2 a_y f(0)^2$ so that when at least one of the parameters δ_z, a_z and $\delta_x + \alpha^2 \delta_y f(0)^2$ is negative, then every point on the equilibrium curve (162) is conditionally unstable. When $\delta_z > 0$, $a_z > 0$, and $\delta_x + \alpha^2 \delta_y f(0)^2 > 0$, based on a simple analysis of the sign of the function $g_3(x, 0)$, we conclude that

- if $b_x > 0$ and $a_x + \alpha^2 a_y f(0)^2 > 0$, then *every* point on the equilibrium curve (162) is conditionally stable;
- if $b_x > 0$ and $a_x + \alpha^2 a_y f(0)^2 \leq 0$, then the points on the equilibrium curve (162) for which $|x| > h_0$ ($|x| < h_0$) with $h_0 = \sqrt{-(a_x + \alpha^2 a_y f(0)^2)/3b_x}$ are conditionally stable (unstable);
- if $b_x < 0$ and $a_x + \alpha^2 a_y f(0)^2 < 0$, then *every* point on the equilibrium curve (162) is conditionally unstable;
- if $b_x < 0$ and $a_x + \alpha^2 a_y f(0)^2 \geq 0$, then the points on the equilibrium curve (162) for which $|x| < h_0$ ($|x| > h_0$) are conditionally stable (unstable).

(2) When $a_z b_z < 0$ and $z = z^{*(i)}$, $i = 1, 2$: From Eq. (156), we require $yf(z^{*(i)}) = -\frac{x(a_x + b_x x^2)}{\alpha a_y}$ for $i = 1, 2$, to obtain the equilibrium states. Thus,

(a) if $f(z^{*(1)}) = 0$, then the points along the line

$$\dot{x} = \dot{y} = \dot{z} = 0,\ x = 0,\ z = z^{*(1)},\ -\infty < y < +\infty \tag{163}$$

are equilibrium states; from Eq. (158) at these points $g_1(z^{*(1)}) = -2a_z$, $g_2(z^{*(1)}) = \delta_x$ and $g_3(0, z^{*(1)}) = a_x$, and it follows that if $\delta_z > 0$, $a_z < 0$, $\delta_x > 0$, and $a_x > 0$ ($\delta_z < 0$ and/or $a_z > 0$ and/or $\delta_x < 0$ and/or $a_x < 0$), then *every* point on the equilibrium line (163) is conditionally stable (unstable).

A similar statement can be made by replacing $z^{*(1)}$ by $z^{*(2)}$ in the previous statement.

(b) if $f(z^{*(1)}) = 0$ and $a_x b_x < 0$, then the points on the lines

$$\dot{x} = \dot{y} = \dot{z} = 0,\ x = x^{*(1)},\ z = z^{*(1)},\ -\infty < y < +\infty \tag{164}$$

$$\dot{x} = \dot{y} = \dot{z} = 0,\ x = x^{*(2)},\ z = z^{*(1)},\ -\infty < y < +\infty \tag{165}$$

are also (additional) nonisolated equilibrium states. For these equilibrium lines, we find $g_1(z^{*(1)}) = -2a_z$, $g_2(z^{*(1)}) = \delta_x$ and $g_3(x^{*(1,2)}, z^{*(1)}) = -2a_x$ so that when $\delta_z > 0$, $a_z < 0$, $\delta_x > 0$ and $a_x < 0$ ($\delta_z < 0$ and/or $a_z > 0$ and/or $\delta_x < 0$ and/or $a_x > 0$), then *every* point on the equilibrium lines (164) and (165) is conditionally

stable (unstable). A similar statement can be made by replacing $z^{*(1)}$ by $z^{*(2)}$ in the previous statement.
On the other hand,

(c) if $f(z^{*(i)}) \neq 0$, $i = 1$, and/or 2, then the points

$$\dot{x}=\dot{y}=\dot{z}=0, z=z^{*(i)}, y=-\frac{x(a_x+b_x x^2)}{\alpha a_y f(z^{*(i)})}, -\infty<x<+\infty \tag{166}$$

form a set of nonisolated equilibrium states; at $y=0$, the equilibrium position is $x=0, y=0, z=z^{*(i)}$, and, if $a_x b_x<0$, then we also have additional equilibrium positions at positions $x = x^{*(j)}$, $y = 0$, $z = z^{*(i)}$, $j = 1, 2$. In this case $g_1(z^{*(i)}) = -2a_z$, $g_2(z^{*(i)}) = \delta_x + \alpha^2\delta_y f(z^{*(i)})^2$ and $g_3(x, z^{*(i)}) = a_x + 3b_x x^2 + \alpha^2 a_y f(z^{*(i)})^2$ so that when $\delta_z < 0$ and/or $a_z > 0$, and/or $\delta_x + \alpha^2\delta_y f(z^{*(i)})^2 < 0$, then *every* point on the equilibrium curve (166) is conditionally unstable.

When $\delta_z > 0$, $a_z < 0$ and $\delta_x + \alpha^2\delta_y f(z^{*(i)})^2 > 0$, similarly to the equilibrium curve described in (1c) above, we find that

- if $b_x > 0$ and $a_x + \alpha^2 a_y f(z^{*(i)})^2 > 0$, then every point on the equilibrium curve (166) is conditionally stable;
- if $b_x > 0$ and $a_x + \alpha^2 a_y f(z^{*(i)})^2 \leq 0$, then the points on the equilibrium curve (166) for which $|x| > h_i$ ($|x| < h_i$) with $h_i = \sqrt{-(a_x + \alpha^2 a_y f(z^{*(i)})^2)/3b_x}$ are conditionally stable (unstable);
- if $b_x < 0$ and $a_x + \alpha^2 a_y f(z^{*(i)})^2 < 0$, then every point on the equilibrium curve (166) is conditionally unstable;
- if $b_x < 0$ and $a_x + \alpha^2 a_y f(z^{*(i)})^2 \geq 0$, then the points on the equilibrium curve (166) for which $|x| < h_0$ ($|x| > h_i$) are conditionally stable (unstable).

Notice that we have obtained infinitely long continua of nonisolated equilibrium positions lying in the plane $z = 0$ and additionally, when $a_z b_z < 0$, in the planes $z = z^{*(1)}$ and $z = z^{*(2)}$.

Example 10.

A. We consider the unconstrained damped system

$$\begin{aligned}\ddot{x} &= y - \dot{x}\\ \ddot{y} &= x - \dot{y}\\ \ddot{z} &= -1 - \dot{z}\end{aligned} \tag{167}$$

The system has no equilibrium positions. We impose on this unconstrained system the constraint $\psi = \dot{y} - z\dot{x} - \dot{z}$. The equation of motion of this constrained system is given, using (5) as:

$$\begin{aligned}\ddot{x} &= [2y + z - xz - 2\dot{x} - z\dot{y} + z\dot{z} - z\dot{x}\dot{z}]/\Delta\\ \ddot{y} &= [-1 - x + yz - xz^2 - (1 + z^2)\dot{y} - \dot{z} - z\dot{x} + \dot{x}\dot{z}]/\Delta\\ \ddot{z} &= [-1 - x - z^2 - yz - \dot{y} - (1 + z^2)\dot{z} + z\dot{x} - \dot{x}\dot{z}]/\Delta\end{aligned} \tag{168}$$

where $\Delta = z^2 + 2$

The equilibrium states of the constrained system, which require that $\dot{x} = \dot{y} = \dot{z} = 0$, are along the line L described by

$$\{(x, y, z, \dot{x}, \dot{y}, \dot{z}):\dot{x} = \dot{y} = \dot{z} = 0, x = -1, y = -z, \quad -\infty < z < \infty\}$$

Upon linearization, the characteristic equation of Eq. (168) at any point $z = \beta$ along the line L is found to be

$$\lambda^2 p(\lambda) = 0, p(\lambda) = \lambda^4 + 2\lambda^3 + (1 + \frac{\beta}{\Delta})\lambda^2 + \frac{\beta}{\Delta}\lambda + \frac{2}{\Delta} \tag{169}$$

The Routh–Hurwitz criterion shows that all the roots of $p(\lambda)$ have real parts that are negative if

$$\beta > 0, \quad \text{and } 2\beta^3 - 7\beta^2 + 4\beta - 16 > 0 \tag{170}$$

which is satisfied when $\beta > \beta^* \cong 3.56789$. If $\beta < \beta^*$, then $p(\lambda)$ has a root with a positive real part. Using the Result, we then conclude that if the point $z > \beta^*$ ($z < \beta^*$) each of the nonisolated equilibrium states on the line L of the constrained system (168) are conditionally stable (unstable).

B. To illustrate the nonintuitive nature of constrained motion, we consider the unconstrained system (167) again, but impose on it the constraint $\psi = \dot{y} - z\dot{x} - \dot{z}^2$, which is nonlinear in the velocity $\dot{z}$. One might intuitively expect that this nonlinearity in the constraint would perhaps lead to more complex behavior, such as additional equilibrium positions, as engendered by the constraint in part A above. Using Eq. (5), the equation of motion of the constrained system is

$$\begin{aligned}\ddot{x} &= [y + xz - \dot{x} + \dot{z}^2(y + z - \dot{x}) - z(\dot{y} - \dot{z} + \dot{x}\dot{z})]/\Delta\\ \ddot{y} &= -[z(\dot{x} - y) + z^2(\dot{y} - x) + \dot{z}(1 - \dot{x}) + \dot{z}^2(1 + \dot{y} - x)]/\Delta\\ \ddot{z} &= -[1 + z^2 + \dot{z}(1 - x + z^2 + yz - \dot{x}z + \dot{x}\dot{z} + \dot{y})]/\Delta\end{aligned} \tag{171}$$

where $\Delta = (1 + z^2 + \dot{z}^2) > 0$. To find the equilibrium states of this system, we require $\dot{x} = \dot{y} = \dot{z} = 0$, so that the right-hand side of third equation above then reduces to -1. Therefore, the system has no equilibrium points! We see that not only is the conventional view incorrect in accepting that the equilibrium points of nonholonomic systems are always nonisolated, but that such systems can have no equilibrium points at all.

Example 11.

We consider the stable unconstrained nonconservative system (99) described in Example 7 subjected to the nonholonomic constraint (100), which is $\psi := \dot{x} - y\dot{z} = 0$. The equations of motion of the constrained system (and its initial conditions) are shown in Eq. (101).

The set of equilibrium states of this system is

$$\{(x, y, z, \dot{x}, \dot{y}, \dot{z}):\dot{x} = \dot{y} = \dot{z} = x = 0, y(1 - 14y - 8z) = 0\} \tag{172}$$

After linearizing Eq. (101) at a point of this set, we obtain the characteristic equation

$$\lambda^2[\lambda^4 + g_1(y)\lambda^2 + g_2(y)] = 0 \tag{173}$$

where $g_1 = y(8 + 7y)/(1 + y^2)$ and $g_2 = 14y^2/(1 + y^2)$.

A point of the equilibrium set requires that $y(1 - 14y - 8z) = 0$ so that the equilibrium states are continuously distributed on the following two lines:

(a) $\dot{x} = \dot{y} = \dot{z} = 0, x = 0, y = 0, \ -\infty < z < +\infty$

At the equilibrium positions at points on this line, all roots of the characteristic equation (173) are equal to zero, since $g_1 = g_2 = 0$, and the stability Result is *not* applicable, showing the gap in our ability to decipher their stability. However, we note that Eq. (101) allows the unbounded solution: $x(t) = y(t) = 0$, $z(t) = rt + c$, where r and c are arbitrary constants; this solution, we emphasize, satisfies the equation of the constraint. It therefore follows that every equilibrium position on the z-axis is conditionally unstable. In particular, the nonholonomic constraint (100) destabilizes the stable equilibrium state at $x = y = z = \dot{x} = \dot{y} = \dot{z} = 0$ of the unconstrained system.

(b) We also have nonisolated equilibria along the line, L, described by:

$$\dot{x} = \dot{y} = \dot{z} = 0, x = 0, z = \tfrac{1}{8} - \tfrac{7}{4}y, \ -\infty < y < +\infty$$

The characteristic equation has a root with positive real part if and only if $y \in D := (-\infty, 0) \cup (8(1 + \sqrt{1 + 1/56}), +\infty)$ and so the system is conditionally unstable along the line L when y belongs in the domain D. A pictorial depiction of the (in)stability of the equilibrium points along L is given in

Fig. 10 showing the y-z plane. The intercept of L along the y-axis is 1/14, and along the z-axis is 1/8. The domain of unstable equilibrium points in L is shown by the dashed black line. In the region shown in red (from point a to point b, see Fig. 10) the nature of stability of the equilibrium points cannot be obtained using our Result since four of the eigenvalues of the linearized equation are purely imaginary at every point in this region. The ordinate of point a, which is on the z-axis, is 1/8, as mentioned, and the abscissa of point b is $8(1+\sqrt{1+1/56}) \approx 16.071$. The z-axis is a line of nonisolated equilibrium points, all of which are unstable. As mentioned, the constrained dynamical system does not have a stable equilibrium point at $x=y=z=\dot{x}=\dot{y}=\dot{z}=0$.

We now consider adding another (holonomic) constraint $\varphi := 15y + 8z = 0$, so that we now have the two constraints:

$$\varphi := 15y + 8z = 0, \quad \text{and } \psi := \dot{x} - y\dot{z} = 0 \tag{174}$$

and particularly show that the addition of this holonomic constraint restores the stability of the equilibrium state $x=y=z=\dot{x}=\dot{y}=\dot{z}=0$ of the unconstrained system, which it had lost when subjected only to the nonholonomic constraint in Eq. (174) (see item (a) above).

Using Eq. (5), the equations of motion of the unconstrained system (99), now subjected to the two constraints (174) are given by

$$\begin{aligned}
\ddot{x} &= \frac{15}{289+225y^2}\left[-15y^2(7x+14y+8z) - 8xy + 15y^2 - \frac{289}{8}\dot{y}^2\right]\\
\ddot{y} &= \frac{8}{289+225y^2}\left[8x + 15y(7x+14y+8z) - 15y - \frac{225}{8}y\dot{y}^2\right]\\
\ddot{z} &= -\frac{15}{289+225y^2}\left[8x + 15y(7x+14y+8z) - 15y - \frac{225}{8}y\dot{y}^2\right]
\end{aligned} \tag{175}$$

with initial conditions belonging to the constraint manifold

$$C = \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^6 : 15y + 8z = 0,\ 15\dot{y} + 8\dot{z} = 0,\ \dot{x} - y\dot{z} = 0\} \tag{176}$$

The set of equilibrium states of this system is

$$\{(x, y, z, \dot{x}, \dot{y}, \dot{z}) : \dot{x} = \dot{y} = \dot{z} = 0,\ z = -15y/8,\ 8x + 15y(7x - y) - 15y = 0\} \tag{177}$$

and in the (x, y)-plane the equilibria are distributed along two branches of the hyperbola $x = 15y(1+y)/(8+105y)$, the first for $y > -8/105$ and the second for $y < -8/105$.

The characteristic equation obtained by the linearizing system (175) about a point of the equilibrium set is

$$\lambda^4\left[\lambda^2 + \frac{(8+105y)^3 + 6208}{7(289+225y^2)(8+105y)}\right] = 0 \tag{178}$$

and it has a root with a positive real part if and only if

$$y \in D_1 := (-4(2+\sqrt[3]{97})/105,\ -8/105) \tag{179}$$

so that the points along the second branch of the equilibrium hyperbola when y belongs in the domain D_1 are unstable. To answer the question of the stability of equilibrium points when

$$y \in D_2 := (-\infty,\ -4(2+\sqrt[3]{97})/105) \cup (-8/105, \infty) \tag{180}$$

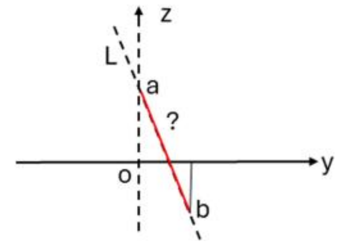

Fig. 10 The z-axis and the line L constitute the set of (nonisolated) equilibrium positions. Positions on the black dashed lines are unstable. The region that cannot be deciphered as being stable or unstable from this analysis lies in between the points a and b along the line L, and is shown by the (red) solid line.

we observe that the dynamics of the system can be described by the equations

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\dfrac{15}{8}y\dot{y} \\ \dot{y} \\ \dfrac{8}{289+225y^2}\left[8x + 15y(7x-y) - 15y\left(1 + \dfrac{15}{8}\dot{y}^2\right)\right] \end{bmatrix} \tag{181}$$

with arbitrary initial conditions, since all orbits of the constrained system lie on the invariant manifold (176). We have replaced z by $-15y/8$, and $\dot{z}$ by $-15\dot{y}/8$ in the last equation above as well as in the equation of the nonholonomic constraint in Eq. (174). It is easy to verify that Eqs. (181) allows the following two first integrals

$$W(x, y, \dot{y}) := \frac{1}{16}(289 + 225y^2)\dot{y}^2 + U(x, y) = h,\ U = 4\left(7x^2 - 2xy + \frac{15}{8}y^2\right) \tag{182}$$

$$F(x, y) := x + \frac{15}{16}y^2 = d \tag{183}$$

where h and d are arbitrary constants.

Let $(x_*, y_*, 0)$ be an equilibrium state of system (181), i.e., $x_* = 15y_*(1+y_*)/(8+105y_*)$, $y_* \neq -8/105$. Then $d_* = F(x_*, y_*) = x_* + 15y_*^2/16$ and the function

$$\hat{U} = U|_{F=d_*} = 4\left(7(d_* - 15y^2/16)^2 - 2y(d_* - 15y^2/16) + \frac{15}{8}y^2\right)$$

The first and second derivatives of the function $\hat{U}(y)$ at the point $y = y_*$, after some algebra, become

$$\frac{\partial \hat{U}}{\partial y}(y_*) = 0 \text{ and } \frac{\partial^2 \hat{U}}{\partial y^2}(y_*) = 8\frac{6208 + (8+105y_*)^3}{448(8+105y_*)}$$

and the second derivative is positive if either $y_* > -8/105$ or $y_* < -4(2+\sqrt[3]{97})/105$. Thus, if $y_* \in D_2$, then the function $\hat{W}(y, \dot{y}) = W|_{F=d_*}$ has a strict minimum at the point $(y_*, 0)$.

Now we introduce the nonnegative function [41]

$$V(x, y, \dot{y}) := [W(x, y, \dot{y}) - W(x_*, y_*, 0)]^2 + [F(x, y) - d_*]^2$$

which has the properties $V(x_*, y_*, 0) = 0$ and $\dot{V}|_{(181)} = 0$. When $y_* \in D_2$ the function V has a strict minimum at the point $(x_*, y_*, 0)$, i.e., it is positive definite with respect to $x - x_*$, $y - y_*$ and $\dot{y}$, since both of the above expressions constituting the function V are nonnegative and the restriction $W|_{F=d_*}$ has a strict minimum at $(y_*, 0)$. Then, according to the Lyapunov stability theorem, when $y_* \in D_2$ the equilibrium state $(x_*, y_*, 0)$ of system (181) is stable (the corresponding equilibrium of the constrained system (175) is conditionally stable with respect to set (176)).

In particular, we see that the equilibrium state at the origin of system (175) is conditionally stable. Thus, adding the holonomic constraint shown in Eq. (174) to the nonholonomic constraint that we had earlier (see part (a)) restores the stability of the origin that the unconstrained system had lost when it was subjected only to this nonholonomic constraint.

Remark 4. The two constraints in Eq. (174), as described before in Example 6B, illustrate that they can be reduced to the two holonomic constraints $\varphi_1 := 15y + 8z = 0$ and $\varphi_2 := x + (15/16)y^2 - c = 0$, where c is an arbitrary constant, which is dependent on the initial conditions. The constrained system can then be reduced to a single degree-of-freedom nonlinear system in which the parameter c can be thought of as a bifurcation parameter, yielding the so-called imperfect pitchfork bifurcation. However, the treatment above is more general since it includes the effects of perturbations in the parameter c, which reflects perturbations in the initial conditions of the system.

More generally, two important observations result from this example. (i) There still remains a gap in our understanding of nonholonomically constrained systems. The central result obtained in this paper is still insufficient in deciphering the stability/instability of equilibrium points in constrained systems, as illustrated by the red region in Fig. 10. (ii) The effect of the addition (removal) of a (set of) constraint(s) to (from) those that a system is subjected to is easy to handle with the use of the explicit equation of constrained motion (5).

Example 12.
We consider here a nonlinear damped unconstrained system described by the equation

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} -k_1x_1 - c_1\dot{x}_1 + x_1^3 \\ -k_2x_2 - c_2\dot{x}_2 + x_1x_3 \\ -k_3x_3 - c_3\dot{x}_1 + x_3^3 \end{bmatrix}, \begin{bmatrix} x_1(0) = a \\ x_2(0) = b \\ x_3(0) = c \end{bmatrix}, \begin{bmatrix} \dot{x}_1(0) = p \\ \dot{x}_2(0) = q \\ \dot{x}_3(0) = r \end{bmatrix} \tag{184}$$

in which, we shall assume, for simplicity, that $k_i, c_i > 0$, $i = 1, 2, 3$, as happens in many mechanical and structural systems. The quantities a, b, c, p, q, r are (real) arbitrary constants.

We subject this nonlinear system to the nonholonomic constraint, which is nonlinear in the velocities, given by

$$\psi := \dot{x}_2 - \frac{1}{2}(\dot{x}_3^2 - \dot{x}_1^2) = 0 \tag{185}$$

The initial condition q in the constrained system is no longer arbitrary; it satisfies the relation $q = (r^2 - p^2)/2$. Using Eq. (5), the equation of motion of the constrained system can be explicitly obtained, and it is shown in Appendix C.

For the unconstrained system to have an equilibrium point, we require $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$, and further, that

$$(x_1^2 - k_1)x_1 = (x_3^2 - k_3)x_3 = x_1x_3 - k_2x_2 = 0$$

Hence, the nine isolated equilibrium points of the unconstrained system are located at

$$\begin{aligned} &x_i = \dot{x}_i = 0, \quad i = 1, 2, 3 \\ &x_1 = \pm\sqrt{k_1}, x_2 = x_3 = 0, \quad \dot{x}_i = 0, \quad i = 1, 2, 3 \\ &x_3 = \pm\sqrt{k_3}, x_1 = x_2 = 0, \quad \dot{x}_i = 0, \quad i = 1, 2, 3 \\ &x_1 = \pm\sqrt{k_1}, x_2 = \frac{\sqrt{k_1k_3}}{k_2}, x_3 = \pm\sqrt{k_3}, \ \dot{x}_i = 0, \quad i = 1, 2, 3, \quad \text{and} \\ &x_1 = \mp\sqrt{k_1}, x_2 = -\frac{\sqrt{k_1k_3}}{k_2}, x_3 = \pm\sqrt{k_3}, \ \dot{x}_i = 0, \quad i = 1, 2, 3 \end{aligned} \tag{186}$$

In each of these relations for the equilibrium points, we take the upper sign consistently for all the three equalities for x_1, x_2, and x_3, or the lower sign for all of them. For example, in the fourth relation, above, we mean that when $x_1, x_3 > 0$, or, when $x_1, x_3 < 0$, then $x_2 > 0$; in the last relation, we mean that when $x_1 < 0$ and $x_3 > 0$ (or vice-versa), then $x_2 < 0$.

After linearizing Eq. (184), the characteristic polynomial at an equilibrium position is

$$(\lambda^2 + c_2\lambda + k_2)(\lambda^2 + c_1\lambda - 3x_1^2 + k_1)(\lambda^2 + c_3\lambda - 3x_3^2 + k_3) \tag{187}$$

It is easy to check that all the equilibrium points are unstable, except for the point at $x_i = \dot{x}_i = 0, \quad i = 1, 2, 3$, which is asymptotically stable.

In the presence of the constraint, as seen from Appendix C, we find that we have lines of nonisolated equilibrium points for $x_2 \in \mathbb{R}$ and $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ that go through the points that satisfy the relations

$$(x_1^2 - k_1)x_1 = (x_3^2 - k_3)x_3 = 0$$

which yield $x_1 = 0$, $x_1 = \pm\sqrt{k_1}$, $x_3 = 0$, $x_3 = \pm\sqrt{k_3}$.

We thus have a total of nine lines of nonisolated equilibrium points parallel to the x_2-axis, each going through these nine points in the (x_1, x_3)-plane. It is as though each of the equilibrium position of the unconstrained system sprouted an infinitely long line of nonisolated equilibrium points parallel to the x_2-axis (see Fig. 11), something that could not have been conceived without having the explicit equations of motion of the constrained system!

After linearization, the characteristic polynomial of the constrained system (see Appendix C) is

$$\lambda^2[\lambda^2 + (c_1 - k_2x_2 + x_1x_3)\lambda + k_1 - 3x_1^2][\lambda^2 + (c_3 + k_2x_2 - x_1x_3)\lambda + k_3 - 3x_3^2] \tag{188}$$

From Eq. (188), the line of nonisolated equilibrium points described by $\{\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = x_1 = x_3 = 0, \ -\infty < x_2 < \infty\}$, therefore, has the characteristic equation

$$\lambda^2[\lambda^2 + (c_1 - k_2x_2)\lambda + k_1 - 3x_1^2][\lambda^2 + (c_3 + k_2x_2)\lambda + k_3] \tag{189}$$

In view of the stability Result, points on this equilibrium line are conditionally stable (unstable) when $x_2 > -c_3/k_2$ and $x_2 < c_1/k_2$ ($x_2 < -c_3/k_2$ or $x_2 > c_1/k_2$). We note that the constrained system behaves independently of the value of the damping coefficient, c_2, somewhat defying intuition, so that the equilibrium positions on this line are stable even when $c_2 < 0$.

In a similar manner, by evaluating the corresponding characteristic polynomial (188) at any point of each of the remaining eight lines of nonisolated equilibrium positions and examining its roots, it is easy to see that the equilibrium positions on each of them are unstable.

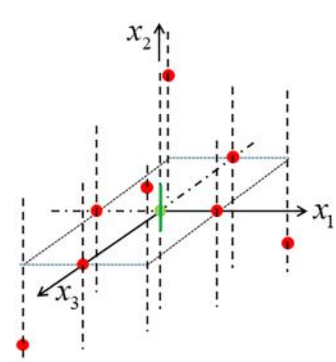

Fig. 11 The solid spheres show the equilibrium positions. All of them are unstable except the equilibrium at the origin. The short, solid line along the x_2-axis shows the nonisolated, conditionally stable equilibrium states of the constrained system. The dashed lines show the unstable nonisolated equilibrium states of the constrained system. The grey rectangle shows the (x_1, x_3)-plane.

5 Conclusions

The motion of constrained systems has been a central problem in mechanics since the time of Lagrange. Over more than two centuries, it has continually attracted the attention of numerous physicists, engineers, and mathematicians, among them, Appell [11], Bloch [42], Boltzmann [43], Chaplygin [44], Chetaev [45], Dirac [46], Lanczos [10], Aiserman and Gatmacher [26], Gauss [47], Hahn [48], Hamel [49], Hertz [21], Korteweg [50], Neimark and Fufaev [24], Rumyantsev [27], and many more. This paper deals with acquiring a deeper understanding of constrained motion that is available today because we now know the general explicit equation of motion of an n degree-of-freedom unconstrained system that is subjected to holonomic and/or nonholonomic constraints.

The paper attempts to assess several of the conventional views that have evolved over the centuries in our understanding of constrained motion. It expands on the knowledge that has so far been gleaned and points out several misunderstandings that have crept into many of our conventional scientific views. We first consider dynamical systems with holonomic constraints and how our current understanding of them differs from the conventional view that has come to be accepted. Next, we consider dynamical systems with nonholonomic constraints, and lastly, the stability of constrained systems in which a new result on the stability analysis of constrained holonomic and/or nonholonomic systems has been developed.

The approach taken in the article is to present this through the use of simple, physically motivated, low-dimensional illustrative examples. They are used to explain the new general features of constrained dynamical systems uncovered through detailed analysis that their low dimensionality permits. We identify precisely where conventional views depart from the truth and dispel several notions that have entered common circulation. As noted, our understanding of constrained motion has benefited considerably from the recent discovery of the general explicit equation of motion, which was previously unavailable to earlier researchers.

The introduction frames the problem of constrained motion and presents the general explicit equation of motion for unconstrained dynamical systems that have nonsingular mass matrices, when subjected to holonomic and/or nonholonomic constraints. The explicit equation is the same for both types of constraints, thereby unifying the treatment of systems that are constrained holonomically and/or nonholonomically. The explicit equation is not restricted to nonholonomic constraints that are in the so-called Pfaffian form—constraints that have almost exclusively been used over the last two centuries—and can also handle constraints that are nonlinear in the velocities. Moreover, they can be used when the prescribed constraints are not independent of one another. This flexibility has considerable value from an applications standpoint, because in complex systems that have a large number of degrees-of-freedom and are subjected to several nonholonomic constraints, it is often difficult to recognize which of them are independent, since the nonholonomic constraints themselves are differential equations.

In the introduction, we prove that the explicit equations of motion for constrained systems are equivalent to Lagrange's equations containing the so-called Lagrange multipliers, which have been used ubiquitously for over two centuries. Unlike the classical Lagrange formulation, however, the explicit equations neither use nor invoke the concept of Lagrange multipliers [13, 14]. Determining these multipliers has long posed a significant challenge for systems with many degrees-of-freedom subject to multiple nonlinear holonomic and/or nonholonomic constraints. As a result, detailed analyses were generally restricted to low-dimensional systems—typically fewer than 10 degrees-of-freedom with fewer than five to seven constraints—rendering the multiplier formulation impractical for most real-world systems, engineered or natural, which often involve hundreds of degrees-of-freedom (if not more) and numerous constraints.

The introduction further derives a general explicit expression for the Lagrange multipliers themselves, representing a substantial advance. It is shown that these multipliers are nonunique when the constraints are dependent, whereas independence of the constraints yields a simpler form.

The second section of the paper presents three simple examples of dynamical systems with holonomic constraints. It re-examines a central conventional notion in analytical dynamics: that an n degree-of-freedom unconstrained system, when subjected to h independent holonomic constraints (each stated as an expression equal to zero), may be thought of as simply an unconstrained $(n-h)$ degree-of-freedom system. By "unconstrained" we mean that the initial velocities can be prescribed arbitrarily.

Contrary to the intuition this statement may suggest, the explicit equation of motion for constrained systems shows that an n degree-of-freedom unconstrained system subjected to h independent holonomic constraints remains an n degree-of-freedom system; it does not become an $(n-h)$ degree-of-freedom system. The conventional notion is therefore, at best, tenuous and, in general, incorrect.

The apparent "reduction" of this n degree-of-freedom system to an $(n-h)$ degree-of-freedom system arises only through a suitable, special one-to-one coordinate transformation in the n-dimensional configuration space. In this transformation, the expressions themselves (equal to zero) that specify each of the h independent holonomic constraints are incorporated into the new coordinate set.

Example 1 considers a nonautonomous holonomic constraint for which such a transformation exists globally over the entire configuration space. It illustrates why the conventional view appears plausible when this global transformation is available. It is shown that corresponding to each constraint, the transformation yields a set of h "shadow" equations—where each shadow coordinate has zero acceleration and zero initial conditions—and a corresponding set of "reduced" (unconstrained) equations in the remaining $(n-h)$ coordinates.

However, when such a transformation is not available globally, the notion that the constrained system may be regarded as having $(n-h)$ degrees-of-freedom requires careful interpretation. Example 2 illustrates this situation. It is shown that depending on the nature of the constraint, a single global one-to-one coordinate transformation may be difficult—or impossible—to obtain. Instead, several coordinate transformations may be required, each valid only locally within a small region of the configuration space. Each local transformation yields a corresponding set—often the same—of shadow equations and an associated set of reduced equations, each valid only within its particular region of the configuration space. Because the reduced equations are generally nonlinear and rarely admit closed-form solutions, determining the system's global behavior becomes challenging, as the local solutions must then be carefully "stitched" together across the boundaries of the respective regions in which they are valid.

These considerations suggest that it may often be conceptually easier and more efficient to work directly with the explicit n degree-of-freedom equations of motion of the constrained system in the original coordinates, rather than attempting coordinate transformations to obtain the $(n-h)$ degree-of-freedom reduced equations. Example 3 presents a constraint for which no coordinate transformation appears capable of yielding reduced equations. In such cases, one must use the explicit equations in the original coordinates, and the constrained n degree-of-freedom system cannot, in any meaningful sense, be reduced to an $(n-h)$ degree-of-freedom system, from a practical standpoint.

Since the explicit equations of motion for constrained systems are nonlinear, computational methods are usually needed to understand the time evolution of the motions that they engender. Example 1 shows that computational results need careful assessment since the n degree-of-freedom equations for the constrained system are unstable. We show that the root cause is that the h "shadow" equations are each unstable. This provides valuable theoretical insights, hitherto unrecognized, into why inaccuracies arise when integrating these equations computationally.

The third section of the paper deals with understanding the behavior of nonholonomic systems. Though simple, the unconstrained systems are selected for their practical relevance, representing types of systems commonly encountered in real life. Example 4 deals with two issues critical to the understanding of the behavior of nonholonomic systems.

First, it shows that the equations of motion of dynamical systems subjected to such constraints cannot be obtained, in general, by using the principle of stationary action. Despite being well known since Hertz's 1894 discovery, this fact appears not to have received the sustained attention it deserves, particularly from physicists. To illustrate this issue, a simple example of a free particle subjected to a nonholonomic constraint is worked out in detail (see Example 4). The example also uncovers some deeper results, namely, that there *are* situations when the principle of stationary action *does* give the correct equations of motion for certain orbits of nonholonomic systems, which we call in this paper "trivial orbits." They arise: (i) when for some special sets of values of initial conditions, the unconstrained dynamical system gives the same orbits as those of the nonholonomic system; being unchanged, this means that these orbits of the unconstrained system are unaffected by the presence of the constraints. This is because these orbits automatically satisfy the constraints, and (ii) when starting from special sets of initial conditions belonging to an invariant manifold of the equations of motion of the nonholonomic system, the nonholonomic constraints become holonomic. But to determine such orbits in (ii), one needs to know the equations of motion of the nonholonomic system, which, indeed, the principle of stationary action cannot, in general, correctly generate.

A second issue that Example 4 illustrates is that, unlike the situation when the constraints are holonomic, the Lagrangian of a nonholonomically constrained system cannot be obtained by inserting the constraint relation in the Lagrangian of the unconstrained system, though a proper Lagrangian for the nonholonomic system might yet exist. Example 5 uses a simple unconstrained potential system subjected to a nonholonomic constraint to illustrate that though the Lagrangians of nonholonomic systems cannot be obtained by inserting the constraints into the Lagrangians of the unconstrained systems upon which the constraints are imposed, the total energy of the unconstrained system could remain conserved even after the imposition of nonholonomic constraints. This unlocks the usefulness of such nonholonomic constraints in areas like quantum mechanics, which to date has concerned itself mainly with holonomic constraints.

Example 6A illustrates the burgeoning of (infinitely many) nonisolated equilibrium states by the imposition of a simple nonholonomic constraint on a simple unconstrained system that has no equilibrium positions at all. Example 6B demonstrates that the inclusion (addition) of one (or more) nonholonomic Pfaffian constraints in a set of holonomic constraints does not necessarily render the holonomic system nonholonomic. Similarly, as shown in Example 6C, a set of Pfaffian nonholonomic constraints does not necessarily lead to a nonholonomic system; at times, it can be reduced to make the set holonomic. While known from Frobeneus's theorem on 1-forms, these facts are often overlooked. Example 6D illustrates a new result, namely, that a set of independent nonholonomic constraints that are nonlinear in the velocities (and to which Frobeneus's theorem no longer applies), can also result in a holonomic system. Example 7 considers an unconstrained system that is linear and nonconservative, of a type encountered in structural and mechanical systems, which is subjected to a nonholonomic constraint. It is used to illustrate some key features in our understanding of nonholonomic systems and the significant gaps in the stability analysis of such systems, examined later.

The last section of the paper examines the stability of constrained systems, with particular emphasis on nonholonomic systems, since significant gaps still remain in our understanding of their stability. It starts with a proof of a new central result applicable to holonomic and/or nonholonomic systems in which the constraints are $\psi_i(q, \dot{q}) = 0$, where $\psi_i(q, \dot{q} = 0) = 0$. This result is applicable to nonholonomic constraints that have nonlinear terms in the velocities ($\dot{q}_j$'s), and it is used extensively in understanding the stability of the constrained systems illustrated in this section. Several of the examples considered in the previous two sections are revisited, and specific prevailing notions related to stability are examined. The conventional view, currently widely prevalent, is that the source of the difficulties in the determination of the stability of nonholonomic systems is that they always have nonisolated equilibrium states, which holonomic systems do not. As pointed out in Example 9A, (see below), all nonholonomic systems do not have nonisolated equilibrium states; some have isolated equilibrium positions. Furthermore, holonomic systems can also have nonisolated equilibrium states, as shown in Example 8.

Example 8A considers the 2-DOF conservative unconstrained system of Example 1 in which the time dependence of the holonomic constraint is removed. The stability of the system is examined. The "reduced" equation for this system, which has a Lagrangian, is analyzed, and its isolated equilibria and their stability are deduced from its potential energy; the equilibria are stationary points of the potential energy, and they are stable if and only if the potential energy is a strict minimum. This standard analysis (the prevalent view) serves as a benchmark for the results that follow. It is shown that an unconstrained system with an indefinite stiffness matrix (that has some negative eigenvalues) can be made stable through the use of holonomic constraints. A seemingly trivial extension of Example 8A considers an analogous 3-DOF conservative unconstrained system with an analogous (single) constraint. The system shows a substantial change in the complexity of its dynamics with the emergence of (infinitely long) nonisolated lines of equilibrium positions. For certain values of the parameters, a circular ring of nonisolated equilibria around the origin emerges in this example, whose stability is analyzed. The stability of the ring is always opposite to that of the equilibrium position at the origin, and with a flip in the sign of the parameters, their stabilities flip, much like a switch. An uncountable infinity of homoclinic orbits leave and enter the origin when it is unstable, and an uncountable infinity of heteroclinic loops appear when each equilibrium position on the ring is unstable, each heteroclinic loop being engendered by two diametrically opposite equilibrium points on the ring. We also illustrate that our geometrical (intuitive) notion of how a "stable" orbit should behave could be very different from the mathematically proven nature of stability.

Example 9A gives a simple example of a 3-DOF nonholonomic system that has a single, isolated equilibrium position. Upon linearization of the constrained system, we show that all six roots of its characteristic equation evaluated at this equilibrium position are zero. This example challenges the common belief that nonholonomic systems necessarily possess nonisolated equilibrium positions. It also rejects the conventional view that it is the nonisolated equilibria that are responsible for the multiple zero roots in the characteristic equation of the linearized equations of motion of a nonholonomic system. The example shows that even an isolated equilibrium position can yield multiple zero roots, and in this case, indeed, all the roots. Example 9B considers the damped, linear unconstrained system introduced in Example 5 and proves stability of the nonisolated fixed points using the result on stability proved herein. It shows that an unstable unconstrained system can be made stable through the use of nonholonomic constraints. Example 9C considers a damped, nonlinear unconstrained system subjected to a nonholonomic constraint, which is a generalization of the one used in Example 9B. A complex stability picture emerges, and nonisolated equilibrium positions along lines and curves in phase space emerge. Here, again, instability of the unconstrained system does not necessarily lead to instability of the constrained system. Our stability result is used to determine the stability of the nonisolated equilibrium points. Example 10A shows how a simple unconstrained unstable system, with no equilibrium position, upon being subjected to a nonholonomic Pfaffian constraint, becomes a constrained system with nonisolated equilibrium positions along an infinitely long line. Our result shows how the character of the equilibrium positions along this line changes, depending on their location along it. A change in the constraint, making it nonlinear in velocities, causes the constrained system in Example 10B to revert to one with no equilibrium points.

Example 11 considers the nonconservative unconstrained system that was introduced before in Example 7, on which a nonholonomic constraint is imposed. It shows that there exist regions in phase space where the equilibrium positions of the constrained dynamical system cannot be definitively determined to be stable or unstable. It shows that the central stability result proved in this paper is, in general, insufficient to ascertain the nature of stability of the equilibrium positions of nonholonomic systems, revealing an important gap that persists in our understanding of the stability of nonholonomic systems. The example also illustrates how readily the stability of equilibrium positions can be reassessed when constraints are added to, or removed from, those originally imposed on a dynamical system. Example 12 considers a nonlinear, damped, unconstrained system subjected to a nonholonomic constraint that is nonlinear in the velocities. The explicit equation of motion of the constrained system is obtained, showing a blossoming, as it were, of infinitely long, parallel lines of nonisolated equilibrium positions from each of the isolated equilibrium positions of the unconstrained system. A detailed stability analysis of the nonisolated equilibrium positions on each line is presented. From the stability result developed in this paper, it is shown that the equilibrium positions on each of the nonisolated equilibrium lines are unstable, except for the equilibrium positions, over a limited stretch, of the line that goes through the origin of the phase space.

This paper shows that the discovery of the explicit equations of motion for constrained systems that have holonomic and/or nonholonomic constraints allows a proper analysis of such systems to be made, including the determination of their equilibrium states and their stability. We demonstrate that several conventional notions regarding constrained motion require refinement or revision, and in some cases, outright rejection. The new stability result developed in this paper, which is applicable to sets of holonomic constraints and/or to sets of nonholonomic constraints, including those that are nonlinear in the velocities, shows that a Lyapunov stability analysis through linearization of the nonlinear explicit equations of motion of constrained systems is now available as a powerful tool in the stability analysis of constrained motion. Through simple examples, the paper points out the quintessential behavior of holonomic and/or nonholonomic systems, and the gaps that remain in our understanding of their stability. The authors hope that the paper will inspire a new generation of researchers to work in this field, which has now been opened up for more systematic investigation.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

Appendix A

We have an unconstrained n-DOF mechanical system on which we impose m holonomic constraints:

$$\varphi_i(x_1, x_2, \ldots, x_n, t) = 0, \; i = 1, 2, \ldots, m \tag{A1}$$

We assume that the constraints are independent.

We seek a local coordinate transformation (mapping) that maps a small domain S_x in $(x_1, x_2, \ldots, x_n; t) := (x; t)$ space to a small domain S_q in the new coordinate space $(q_1, q_2, \ldots, q_n; t) := (q; t)$. That is, we want the mapping

$$(x_1, x_2, \ldots, x_n, t) \to (q_1, q_2, \ldots q_m, q_{m+1}, \ldots, q_n, t) \tag{A2}$$

The coordinates $(q_1, q_2, \ldots, q_n, t)$ will be called the "new" coordinates in what follows. We set the new coordinates in this mapping so that

$$q_i = \varphi_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, m \tag{A3}$$

Our mapping Eq. (A2) now reads

$$(x_1, x_2, \ldots, x_n, t) \to (\varphi_1, \varphi_2, \ldots \varphi_m, q_{m+1}, \ldots, q_n, t) \tag{A4}$$

where we have suppressed the arguments of the m φ_i's shown in Eq. (A3). We now choose the coordinates $q_{m+1}, \ldots, q_n$ so that

$$\det[J] := \det\left[\frac{\partial(\varphi_1, \varphi_2, \ldots, \varphi_m, q_{m+1}, q_{m+2}, \ldots, q_n)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_n)}\right] \neq 0 \tag{A5}$$

so that we have a one-to-one mapping from the small local domain S_x in $(x; t)$ to the new local domain S_q in $(q; t)$. The new coordinates $q_{m+1}, \ldots, q_n$ are independent of each of the others in the new coordinate space $(q_1, q_2, \ldots, q_n, t)$. In fact, now the coordinates $q_1, q_2, \ldots q_m, q_{m+1}, \ldots, q_n; t$ are all independent of one another.

Since the system is constrained, using Eqs. (A1) and (A3), our mapping Eq. (A4) now becomes

$$(x_1, x_2, \ldots, x_n, t) \to (\underbrace{0, 0, \ldots 0}_{m}, q_{m+1}, q_{m+2}, \ldots, q_n, t) \tag{A6}$$

and therefore, from our one to-one mapping Eq. (A6), that

$$x_i = f(q_{m+1}, q_{m+2}, \ldots q_n, t), \quad i = 1, 2, \ldots, n \tag{A7}$$

Since the constraints $\varphi_i = 0, \quad i = 1, 2, \ldots, m$ are satisfied at every instant of time, $q_i \equiv 0, \quad i = 1, 2, \ldots, m$, for all time, and the dynamical equations of motion of the constrained system for the first m local coordinates in the new configuration space are simply

$$\ddot{q}_i(t) = 0, \quad i = 1, 2, \ldots, m \tag{A8}$$

The remainder of the $(n - m)$ equations of motion of the constrained system involves only the new local coordinates $q_{m+1}, q_{m+2}, \ldots, q_n$ that are chosen, as stated in Eq. (A5), to satisfy

$$\det[J] := \det\left[\frac{\partial(q_1, q_2, \ldots, q_m, q_{m+1}, q_{m+2}, \ldots, q_n)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_n)}\right] \neq 0$$

Given the constraints $\varphi_i = 0, \quad i = 1, 2, \ldots, m$, in Eq. (A1), the virtual displacement column vector $\delta x := [\delta x_1, \delta x_2, \ldots, \delta x_n]^T$ satisfies the relation

$$A\delta x := \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1}, & \frac{\partial \varphi_1}{\partial x_2}, & \ldots\ldots, & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1}, & \frac{\partial \varphi_2}{\partial x_2}, & \ldots\ldots, & \frac{\partial \varphi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_m}{\partial x_1}, & \frac{\partial \varphi_m}{\partial x_2}, & \ldots\ldots, & \frac{\partial \varphi_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \ldots \\ \ldots \\ \delta x_n \end{bmatrix} = 0 \tag{A9}$$

Note that the matrix A is m by n.

We now explore whether the equations of motion of the constrained system, which involve the remaining $(n - m)$ coordinates $q_{m+1}, q_{m+2}, \ldots, q_n$, represent a constrained or an unconstrained subsystem. The subsystem would be unconstrained if the virtual displacements, $\delta q_{m+1}, \delta q_{m+2}, \ldots, \delta q_n$, can be arbitrarily chosen. Our question then becomes: Can these virtual displacements $\delta q_{m+1}, \delta q_{m+2}, \ldots, \delta q_n$ be chosen arbitrarily when the x_i's are given by Eq. (A7)? That is, would the equation

$$A\delta x = 0$$

still be satisfied while assigning arbitrary values for $\delta q_{m+1}, \delta q_{m+2}, \ldots, \delta q_n$? We proceed as follows.

Since the first m new coordinates $q_i = \varphi_i$ (see Eq. (A3)), relation (A9) can be written as:

$$A\delta x := \begin{bmatrix} \frac{\partial q_1}{\partial x_1}, \frac{\partial q_1}{\partial x_2}, \ldots\ldots, \frac{\partial q_1}{\partial x_n} \\ \frac{\partial q_2}{\partial x_1}, \frac{\partial q_2}{\partial x_2}, \ldots\ldots, \frac{\partial q_2}{\partial x_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial q_m}{\partial x_1}, \frac{\partial q_m}{\partial x_2}, \ldots\ldots, \frac{\partial q_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \ldots \\ \ldots \\ \delta x_n \end{bmatrix} = 0 \tag{A10}$$

Furthermore, from Eq. (A7) we see that

$$dx_i = \frac{\partial x_i}{\partial q_{m+1}} dq_{m+1} + \frac{\partial x_i}{\partial q_{m+2}} dq_{m+2} + \cdots + \frac{\partial x_i}{\partial q_n} dq_n + \frac{\partial x_i}{\partial t} dt,$$
$$i = 1, 2, \ldots, n$$

so that the virtual displacement δx_i is obtained (by "freezing" time) as:

$$\delta x_i = \frac{\partial x_i}{\partial q_{m+1}} \delta q_{m+1} + \frac{\partial x_i}{\partial q_{m+2}} \delta q_{m+2} + \cdots + \frac{\partial x_i}{\partial q_n} \delta q_n, \quad i = 1, 2, \ldots, n \tag{A11}$$

Now consider the product $A\delta x$, and consider this product when the δx_i are given by Eq. (A11). We get

$$A\delta x = \begin{bmatrix} \frac{\partial q_1}{\partial x_1}, \frac{\partial q_1}{\partial x_2}, \ldots\ldots\ldots\ldots, \frac{\partial q_1}{\partial x_n} \\ \frac{\partial q_2}{\partial x_1}, \frac{\partial q_2}{\partial x_2}, \ldots\ldots\ldots\ldots, \frac{\partial q_2}{\partial x_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial q_m}{\partial x_1}, \frac{\partial q_m}{\partial x_2}, \ldots\ldots\ldots\ldots, \frac{\partial q_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial q_{m+1}}, \frac{\partial x_1}{\partial q_{m+2}}, \ldots, \frac{\partial x_i}{\partial q_n} \\ \frac{\partial x_2}{\partial q_{m+1}}, \frac{\partial x_2}{\partial q_{m+2}}, \ldots, \frac{\partial x_2}{\partial q_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial x_n}{\partial q_{m+1}}, \frac{\partial x_n}{\partial q_{m+2}}, \ldots, \frac{\partial x_n}{\partial q_n} \end{bmatrix} \begin{bmatrix} \delta q_{m+1} \\ \delta q_{m+2} \\ \ldots \\ \ldots \\ \delta q_n \end{bmatrix} := B \begin{bmatrix} \delta q_{m+1} \\ \delta q_{m+2} \\ \ldots \\ \ldots \\ \delta q_n \end{bmatrix} \tag{A12}$$

The product of the two matrices above is denoted by the m by $(n-m)$ matrix B. We consider now the (i, j)-element of B; here, $1 \le i \le m$, and $m+1 \le j \le n$. Doing the standard ith row by the jth column multiplication, we find that

$$[B]_{i,j} = \frac{\partial q_i}{\partial x_1}\frac{\partial x_1}{\partial q_j} + \frac{\partial q_i}{\partial x_2}\frac{\partial x_2}{\partial q_j} + \cdots + \frac{\partial q_i}{\partial x_n}\frac{\partial x_n}{\partial q_j} = \frac{\partial q_i}{\partial q_j} = 0,$$
$$\text{for } 1 \le i \le m, \quad m+1 \le j \le n,$$

where the last equality follows from the fact that q_i, $1 \le i \le m$ and q_j, $m+1 \le j \le n$ are independent coordinates. Hence, every element of the matrix B is zero and

$$A\delta x = B \begin{bmatrix} \delta q_{m+1} \\ \delta q_{m+2} \\ \ldots \\ \ldots \\ \delta q_n \end{bmatrix} = 0 \tag{A13}$$

since the matrix $B = 0$. The most important observation from Eq. (A13) is that $A\delta x = 0$ for *arbitrary* values of the $(n-m)$ virtual displacements $\delta q_{m+1}, \delta q_{m+2}, \ldots, \delta q_n$, since it is the matrix B, which (automatically) becomes zero, that causes this. We then come to following important result.

Theorem. When we have: (i) an n-DOF unconstrained system (ii) impose on this system the m independent holonomic constraints (A1), (iii) perform the local coordinate change (A2), (iv) set the first m new coordinates to $q_i = \varphi_i$, $i = 1, 2, \ldots, m$, and (v) determine the remaining $(n-m)$ new coordinates $q_{m+1}, q_{m+2}, \ldots, q_n$ so that

$$\det[J] := \det\left[\frac{\partial(q_1, q_2, \ldots, q_m, q_{m+1}, q_{m+2}, \ldots, q_n)}{\partial(x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_n)}\right] \ne 0$$

then the first m equations of motion of the constrained n-DOF system in the new coordinates become

$$\ddot{q}_i(t) = 0, \quad i = 1, 2, \ldots, m$$

and the remainder of the equations of motion describing the constrained n-DOF motion, which (only) use the $(n-m)$ coordinates $q_{m+1}, q_{m+2}, \ldots, q_n$, becomes an *unconstrained* subsystem, since the virtual displacements in these coordinates can be arbitrarily prescribed, while preserving the restriction $A\delta x = 0$, where the x_i are functions of (only) the remaining $(n-m)$ coordinates, $q_{m+1}, q_{m+2}, \ldots, q_n$.

Appendix B

We begin with Eq. (78), which is

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}} - \frac{\partial L^*}{\partial x} = z\dot{x}\dot{z}, \quad \text{and} \quad \frac{d}{dt}\frac{\partial L^*}{\partial \dot{z}} - \frac{\partial L^*}{\partial z} = -z\dot{x}^2 \tag{B1}$$

We consider a change in the independent variable $t \to \tau$, defined by $d\tau = N(z(t))dt$, where the coordinate function $N(z(t))$ is subject to determination. For brevity, in what follows we will suppress the argument $z(t)$ in $N(z(t))$, unless needed for clarity. This change in the independent variable gives the following identities:

$$\dot{x} = x'N(z) \text{ and } \dot{z} = z'N(z)$$

where the primes denote differentiations with respect to τ. Recall that

$$L^* = \frac{1}{2}[(1 + z(t)^2)\dot{x}^2 + \dot{z}^2] = \frac{N(z)^2}{2}[(1 + z^2)x'^2 + z'^2] := \hat{L} \tag{B2}$$

so that $\frac{\partial L^*}{\partial x} = \frac{\partial \hat{L}}{\partial x} = 0$. Furthermore,

$$\frac{\partial L^*}{\partial z} = \dot{x}^2 z = x'^2 z N^2, \text{ and } \frac{\partial \hat{L}}{\partial z} = x'^2 z N^2 + [(1 + z^2)x'^2 + z'^2]N\frac{dN}{dz}$$

so that

$$\frac{\partial L^*}{\partial z}=\frac{\partial \hat{L}}{\partial z}-[(1+z^2)x'^2+z'^2]N\frac{dN}{dz} \tag{B3}$$

We also have from Eq. (B2) and the identities that

$$\begin{aligned}\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}}&=\frac{d[(1+z^2)\dot{x}]}{dt}=\frac{d[(1+z^2)x'N(z)]}{dt}\\&=N\frac{d[(1+z^2)x'N(z)]}{d\tau}\\&=(1+z^2)x''N^2+2zz'x'N^2+(1+z^2)x'z'N\frac{dN}{dz}\end{aligned} \tag{B4}$$

and

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial x'}&=\frac{d[N^2(1+z^2)x']}{d\tau}=(1+z^2)x''N^2+2zz'x'N^2\\&+2(1+z^2)x'z'N\frac{dN}{dz}\end{aligned} \tag{B5}$$

Noting that $\frac{\partial L^*}{\partial x}=0$, from Eqs. (B4) and (B5) we deduce that

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}}-\frac{\partial L^*}{\partial x}=\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial x'}-(1+z^2)x'z'N\frac{dN}{dz}=z\dot{x}\dot{z}=zx'z'N^2$$

in which the third equality comes from Eq. (B1). Since $\frac{\partial \hat{L}}{\partial x}=0$, this gives

$$\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial x'}-\frac{\partial \hat{L}}{\partial x}=zx'z'N^2+(1+z^2)x'z'N\frac{dN}{dz}$$

or

$$\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial x'}-\frac{\partial \hat{L}}{\partial x'}=x'z'N\left[Nz+(1+z^2)\frac{dN}{dz}\right] \tag{B6}$$

Similarly, we find that

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{z}}=\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial z'}-z'^2N\frac{dN}{dz}$$

and by using (B3), we obtain,

$$\begin{aligned}\frac{d}{dt}\frac{\partial L^*}{\partial \dot{z}}-\frac{\partial L^*}{\partial z}&=\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial z'}-z'^2N\frac{dN}{dz}-\left[\frac{\partial \hat{L}}{\partial z}-[(1+z^2)x'^2+z'^2]N\frac{dN}{dz}\right]\\&=-zx'^2N^2\end{aligned}$$

where the last equality comes from Eq. (B1). This gives

$$\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial z'}-\frac{\partial \hat{L}}{\partial z}=-x'^2N\left[Nz+(1+z^2)\frac{dN}{dz}\right] \tag{B7}$$

Our transformation from $t\to\tau$ yields the two Eqs. (B6) and (B7) both of which have a common factor on their right-hand sides shown by the square brackets.

On putting

$$Nz+(1+z^2)\frac{dN}{dz}=0 \tag{B8}$$

we find that

$$N(z)=\frac{1}{\sqrt{1+z^2}} \tag{B9}$$

is a solution to Eq. (B8). The function $N(z)$ in Eq. (B9) then reduces Eqs. (B6) and (B7) to

$$\frac{d}{d\tau}\frac{\partial \hat{L}}{\partial x'}-\frac{\partial \hat{L}}{\partial x'}=0 \text{ and } \frac{d}{d\tau}\frac{\partial \hat{L}}{\partial z'}-\frac{\partial \hat{L}}{\partial z}=0$$

which are the Lagrange equations of motion in terms of the "scaled" time τ. We have thus found the transformation $d\tau=N(z)dt=\frac{dt}{\sqrt{1+z^2}}$ that causes

$$\hat{L}=\frac{N(z)^2}{2}[(1+z^2)x'^2+z'^2]=\frac{1}{2}\left[x'^2+\frac{1}{1+z^2}z'^2\right] \tag{B10}$$

to be the Lagrangian for which the Lagrange equations give the correct equations of motion of the constrained system.

Appendix C:

Using Eq. (5) in the paper, the equation of motion of the constrained system is found to be

$$\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\\\ddot{x}_3\end{bmatrix}=\frac{1}{1+\dot{x}_1^2+\dot{x}_3^2}\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix} \tag{C1}$$

where

$$\begin{aligned}F_1=&-(k_1x_1+c_1\dot{x}_1)(1+\dot{x}_3^2)+\dot{x}_1(k_2x_2-k_3x_3\dot{x}_3)\\&+\dot{x}_1(c_2\dot{x}_2-c_3\dot{x}_3^2)+x_1^3(1+\dot{x}_3^2)+x_3\dot{x}_1(x_3^2\dot{x}_3-x_1)\end{aligned} \tag{C2}$$

$$\begin{aligned}F_2=&k_1x_1\dot{x}_1-k_2x_2(\dot{x}_1^2+\dot{x}_3^2)-k_3x_3\dot{x}_3\\&+c_1\dot{x}_1^2-c_2\dot{x}_2(\dot{x}_1^2+\dot{x}_3^2)-c_3\dot{x}_3^2+x_1x_3(\dot{x}_1^2+\dot{x}_3^2)-x_1^3\dot{x}_1+x_3^3\dot{x}_3\end{aligned} \tag{C3}$$

$$\begin{aligned}F_3=&-\dot{x}_3(k_2x_2+k_1x_1\dot{x}_1)-(k_3x_3+c_3\dot{x}_3)(1+\dot{x}_1^2)\\&-\dot{x}_3(c_2\dot{x}_2+c_1\dot{x}_1^2)+x_3^3(1+\dot{x}_1^2)+x_1\dot{x}_3(x_1^2\dot{x}_1+x_3)\end{aligned} \tag{C4}$$

We note that at the equilibrium states we require $\dot{x}_i=0$, $i=1, 2, 3$. Furthermore, at $\dot{x}_1=\dot{x}_2=\dot{x}_3=0$, we see from Eqs. (C2)–(C4) that $F_1=-k_1x_1+x_1^3$, $F_2=0$, $F_3=-k_3x_3+x_3^3$, so that the set of equilibrium states of the constrained system is

$$\begin{aligned}&\{(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3):\dot{x}_1=\dot{x}_2=\dot{x}_3=0,\ -k_1x_1+x_1^3=0,\\&-k_3x_3+x_3^3=0\}\end{aligned} \tag{C5}$$

The matrices $(\partial\tilde{F}/\partial x)$ and $(\partial\tilde{F}/\partial\dot{x})$, where $\tilde{F}$ denotes the right-hand side of Eq. (C1), evaluated at $\dot{x}_1=\dot{x}_2=\dot{x}_3=0$, are given by

$$\left(\frac{\partial\tilde{F}}{\partial x}\right)=\mathrm{diag}(-k_1+3x_1^2,\ 0,\ -k_3+3x_3^2) \tag{C6}$$

and

$$\left(\frac{\partial\tilde{F}}{\partial\dot{x}}\right)=\begin{bmatrix}-c_1+k_2x_2-x_1x_3 & 0 & 0\\k_1x_1-x_1^3 & 0 & -k_3x_3+x_3^3\\0 & 0 & -c_3-k_2x_2+x_1x_3\end{bmatrix} \tag{C7}$$

so that the characteristic equation $\det[\lambda^2I_3-\lambda(\partial\tilde{F}/\partial\dot{x})-(\partial\tilde{F}/\partial x)]=0$ becomes

$$\begin{aligned}&\lambda^2[\lambda^2+(c_1-k_2x_2+x_1x_3)\lambda+k_1-3x_1^2]\times\\&[\lambda^2+(c_3+k_2x_2-x_1x_3)\lambda+k_3-3x_3^2]=0\end{aligned} \tag{C8}$$

Appendix D:

The Lyapunov–Malkin theorem [34,35]

Consider the dynamical system

$$\begin{aligned}&\dot{z}=Bz+Z(y, z),\quad y\in R^m,\quad z\in R^k\\&\dot{y}=Y(y, z)\end{aligned}$$

where B is a k by k matrix and $Y(y, z)$, $Z(y, z)$ represent nonlinear terms. If $Y(y, 0)=0$, $Z(y, 0)=0$ and all the eigenvalues of the matrix B have negative real parts, then the equilibrium $y=0$, $z=0$ of the dynamical system is stable with respect to the coordinates y and z, and asymptotically stable with respect to z. If a solution $(y(t), z(t))$ of the dynamical system is sufficiently close to the solution $(0, 0)$, then $\lim_{t\to\infty}z(t)=0$ and $\lim_{t\to\infty}y(t)=c$, where c is a constant.

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